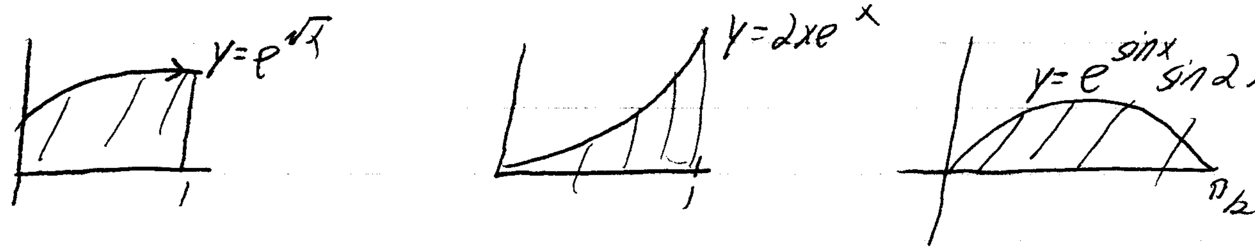
Lecture 2

Review u substitution can be used for definite or indefinite integrals to transform problem into antiderivatives that we "know."

Ex $\int \frac{\cos x}{\sin^2 x}\,dx$ $\quad u=\sin x \quad du=\cos x\,dx$

$\int \frac{1}{u^2}\,du = -\frac{1}{u}+C = -\frac{1}{\sin x}+C$

Ex #75. Which of the following are equal areas?



$A1=\int_0^1 e^{\sqrt{x}}\,dx$ $\quad u=\sqrt{x} \quad du=\frac{1}{2\sqrt{x}}\,dx$

so $dx=2\sqrt{x}\,du=2u\,du$

$A=\int_0^1 2ue^{u}\,du$
$\parallel$
$A2$

$A3 \quad u=\sin x \quad du=\cos x\,dx \qquad \sin 2x=2\sin x\cos x$
$=2u\,du$

$A3=\int_0^1 2ue^{u}\,du$

All 3 are equal!

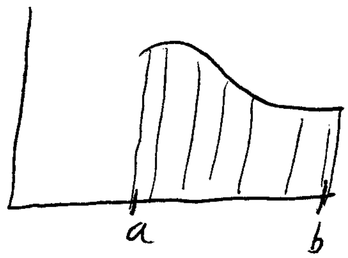
Ex $\int \frac{\tan^{-1}x}{1+x^2}\,dx \quad u=\tan^{-1}x \quad du=\frac{1}{1+x^2}\,dx$

$=\frac{1}{2}(\tan^{-1}x)^2$

6.1 Areas Between Curves

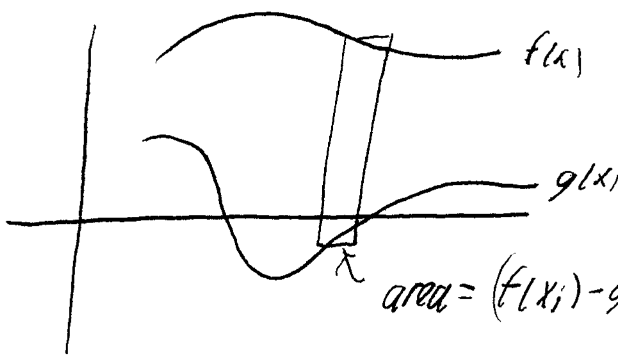
Review Suppose $f(x) \geq 0$ on $[a,b]$. Then

$\int_a^b f(x)\,dx$ is the area under $f(x)$ and above x-axis



since $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ limit of areas of rectangles

Generalize Suppose $g(x) \leq f(x)$ for x in $[a,b]$



area $= (f(x_i) - g(x_i))\,\Delta x$

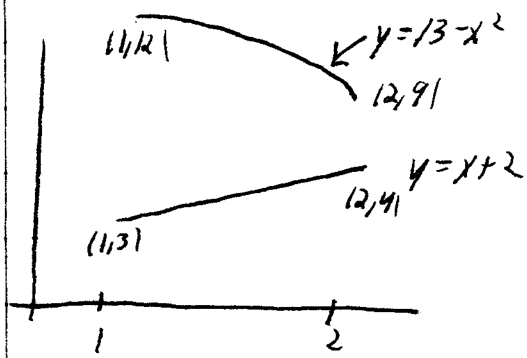
Do limit to get

Thm If $g(x) \leq f(x)$ for x in $[a,b]$ then the area between $y = g(x)$, $y = f(x)$ and lines $x = a$ and $x = b$ is

$$\int_a^b f(x) - g(x)\,dx$$

Picture

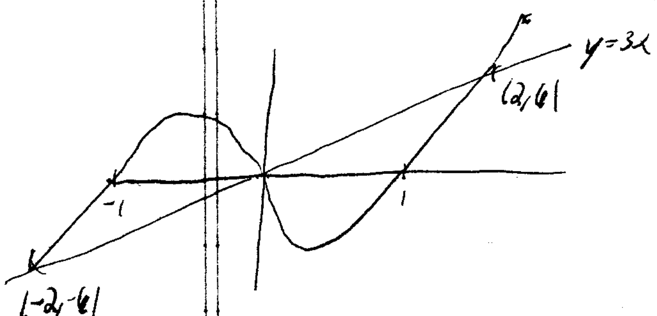
Ex Find area between $y=13-x^2$ and $y=x+2$ above $[1,2]$



$$A = \int_1^2 13-x^2-(x+2) = \int_1^2 11-x^2-x$$

$$= 11x - x^3/3 - x^2/2 \Big|_1^2 = 43/6$$

Ex Find area enclosed by $y=x^3-x$ and $y=3x$



<u>Step 1</u>: Find intersections

$x^3-x=3x$

$x^3-4x=0$

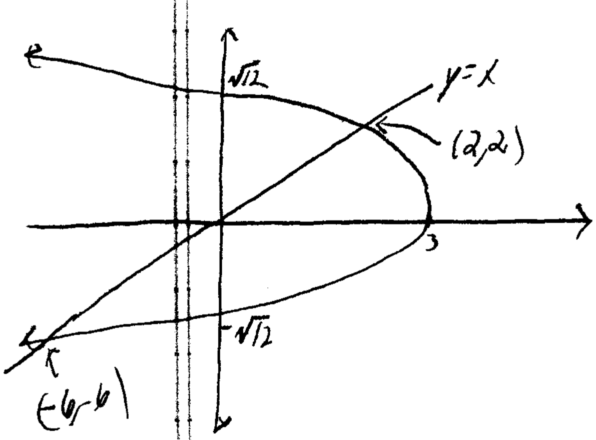
$x(x+2)(x-2)=0$

$$\text{Area} = \int_{-2}^{0} x^3-x-3x\,dx + \int_0^2 3x-(x^3-x)\,dx$$

$$= x^4/4 - 2x^2 \Big|_{-2}^{0} + 2x^2 - x^4/4 \Big|_0^2$$

$$= 4+4 = \boxed{8}$$

Ex Find area enclosed by $y=x$ and $4x+y^2=12$.



$4x+x^2=12$

$x^2+4x-12=0 \qquad (x+6)(x-2)$

$x=2,-6$

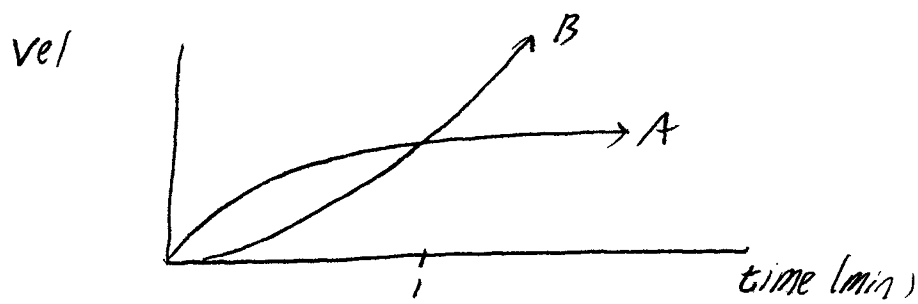
Easier to turn on side, find area between $x=y$ and $x=3-y^2/4$

$$\text{Area} = \int_{-6}^{2} 3-y^2/4 - y \, dy = 3y - y^3/12 - y^2/2 \Big|_{-6}^{2}$$
$$= (6 - 2/3 - 2) - (-18 + 18 - 18)$$
$$= (10/3) - (-18) = 64/3$$

Rmk In general the area between $y=f(x)$ and $y=g(x)$ between $x=a$ and $x=b$ is

$$\int_a^b |f(x)-g(x)| \, dx$$

Ex Two cars start side by side, graph shows velocity:



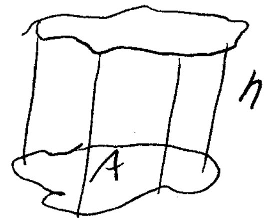
1. Which car is ahead after 1 minute?

2. What does shaded area represent?

Volume

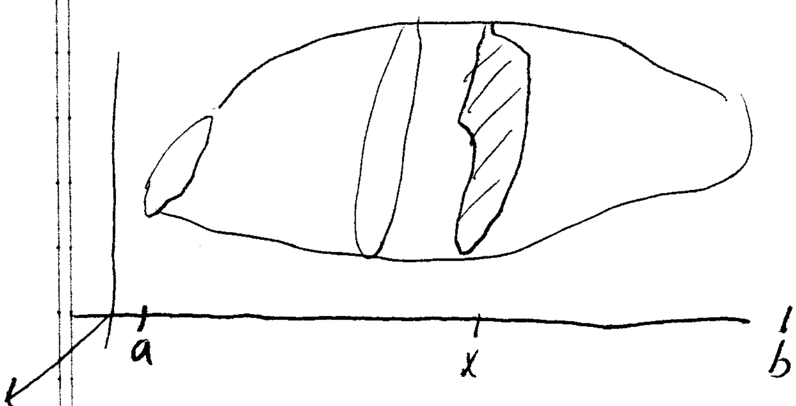
Recall We "knew" area of a rectangle, we defined other areas by approximating w/ rectangles and taking limits to obtain integral

Similarly for a cylinder



Volume = $\text{Area}_{\text{base}} \cdot \text{height}$

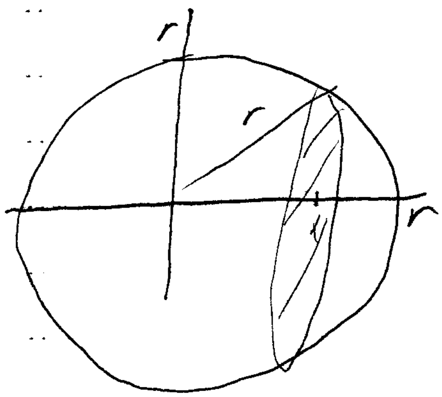
For more complicated shapes we again will take approximation by cylinders then limit to obtain an integral.



Suppose S is a solid lying between $x=a$ and $x=b$. Let $A(x)$ be the area of the cross section cut out by plane $\perp$ to x axis. Then

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

<u>Ex</u> Find volume of sphere of radius r.



cross section is a circle of radius $y = \sqrt{r^2 - x^2}$

So area of cross section is $\pi(r^2 - x^2)$

$$\text{Volume} = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi r^2 x - \tfrac{\pi}{3} x^3 \Big|_{-r}^{r}$$

$$= \left(\pi r^3 - \tfrac{\pi}{3} r^3\right) - \left(-\pi r^3 + \tfrac{\pi}{3} r^3\right)$$

$$= \tfrac{4}{3} \pi r^3$$