Lecture 19

Review Series $a_1 + a_2 + \dots = \sum_{n=1}^{\infty} a_i$, take sequence of partial sums:

$$S_n = a_1 + a_2 + \dots + a_n. \quad (\text{Note: } a_n = S_n - S_{n-1})$$

Def $\sum_{n=1}^{\infty} a_n = S$ if $\lim_{n\to\infty} S_n = S$.

Example Geometric Series $a + ar + ar^2 + \dots = \sum_{n=1}^{\infty} ar^{n-1}$
converges to $\frac{a}{1-r}$ if $|r|<1$, diverges otherwise.

Ex Harmonic series $\sum \frac{1}{n}$ diverges.

Ex/Thm $\sum a_n$ can't possibly converge unless $\lim_{n\to\infty} a_n = 0$.

Ex $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} + \frac{5}{720}$

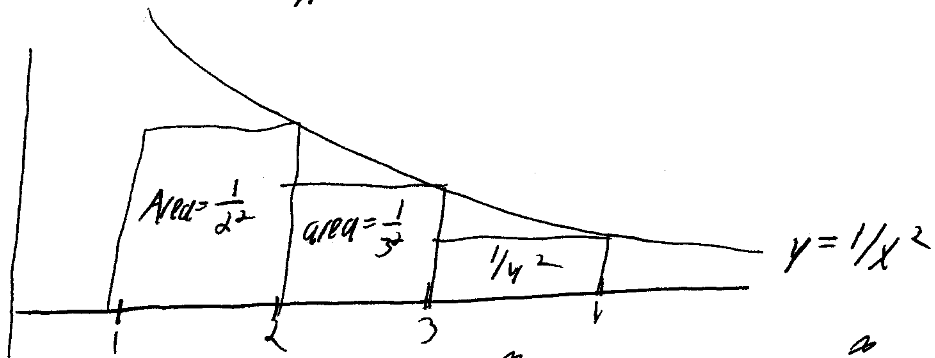
$S_1 = 1/2 \quad S_2 = 5/6 \quad S_3 = 23/24 \quad S_4 = \frac{119}{120}$ Check $S_n = \frac{n!-1}{n!}$ by induction

so $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

11.3 Integral Test and Estimates of Sums

Remark Only in special cases (geom series, telescoping series) do we get formula for S_n, need other techniques to show convergence, estimate $\sum a_n$.

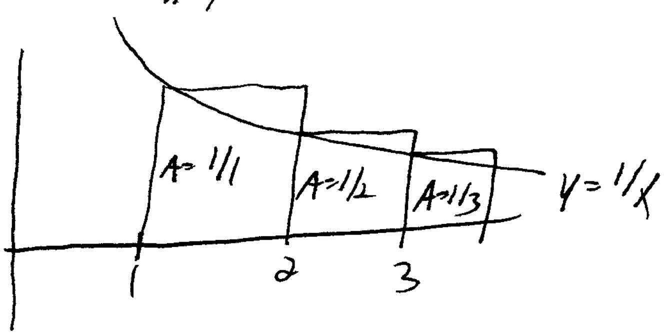
Example I claim $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and is < 2 (actually $= \pi^2/6$)



Clearly $\sum_{n=2}^{\infty} 1/n^2 < \int_1^{\infty} 1/x^2\,dx = -\frac{1}{x}\Big|_1^{\infty} = 0 - -1 = 1$

Thus $\sum_{n=1}^{\infty} 1/n^2 < 1 + 1 = 2.$

Example I claim $\sum_{n=1}^{\infty} 1/n$ diverges.



Clearly $0 < \int_1^{\infty} 1/x\,dx < \sum_{n=1}^{\infty} 1/n$ but $\int_1^{\infty} 1/x\,dx = \infty$

so $\sum_{n=1}^{\infty} 1/n$ diverges

Thm (Integral Test) Suppose $f(x)$ is continuous positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

$\sum_{n=1}^{\infty} a_n$ converges if & only if $\int_1^{\infty} f(x)\,dx$ converges

Remarks 1. $\sum_{n=4}^{\infty} f(n)$ etc. works, e.g. $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ converges $\leftrightarrow$ $\int_4^{\infty} 1/(x-3)^2 \, dx$ does.

2. Only need $f(n)$ eventually decreasing, since changing beginning terms of series doesn't change convergence.

Ex. Does $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converge?

$$= \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \dots$$

$f(x) = \frac{1}{x^2+1}$ is continuous, decreasing and $\int_2^{\infty} \frac{1}{x^2+1} \, dx = \tan^{-1} x \Big|_2^{\infty}$
$= \pi/2 - \tan^{-1} 2$

Yes.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

called p-series.

$p = 3$, $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots$ converges

OPEN PROBLEM: TO WHAT? (open for p odd)

Ex. Show $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges

Proof $f(x) = \frac{\ln x}{x}$ $f'(x) = \frac{1 - \ln x}{x^2}$ is < 0 for $x > e$

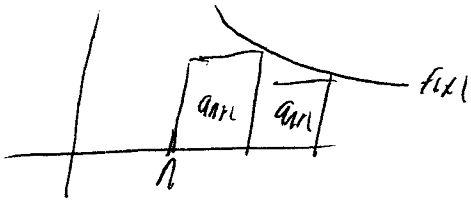
so decreasing eventually

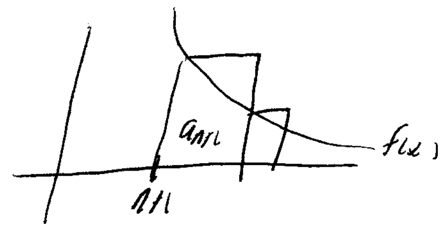
$$\int_1^{\infty} \frac{\ln x}{x} \, dx = \tfrac{1}{2}(\ln x)^2 \Big|_1^{\infty} \quad \text{DIVERGES}$$

Remainder Estimate Suppose f continuous, positive, decreasing for $x \geq n$.
Suppose $\sum a_n = s$ converges. Then:

$$\text{If } R_n = s - s_n \text{ then } \int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Proof $R_n = a_{n+1} + a_{n+2} + \ldots$



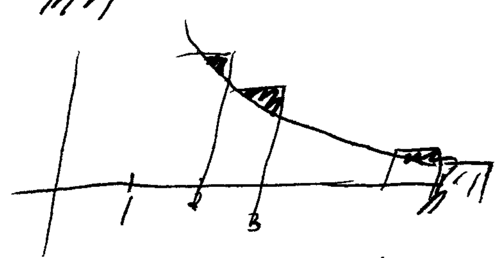


Problems

1. Find s_{10} for series $\sum_{n=1}^{\infty} 1/n^4$, estimate error
   Find value of n so s_n is within 0.00001

   Rmk: $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$

2. Let $t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$



t_n = red area

Notice that $t_n > 0$

Check $t_n - t_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1}$, show > 0 (Differentiate)

COR t_n is non-decreasing, bounded below, converges to

γ = Euler Constant OPEN PROBLEM: Is γ rational?

IF SO, DENOM IS $> 10^{242080}$

Comparison Tests   Suppose $\sum a_n, \sum b_n$ series w/ positive terms.

1. If $a_n \le b_n$ for all n and $\sum b_n$ converges then $\sum a_n$ converges.

2. If $b_n \le a_n$ for all n and $\sum b_n$ diverges then $\sum a_n$ diverges

Use to compare against:
- p series $\sum 1/n^p$ converges iff only if $p>1$
- geom $\sum ar^{n-1}$ convs iff only if $|r|<1$

EX
1. Show $\sum_{n=1}^{\infty} \frac{6}{2^n+n}$ converges
2. Show $\sum_{n=1}^{\infty} \frac{\ln(n)^3}{n}$ diverges

Limit Comparison Test   Suppose $\sum a_n, \sum b_n$ series w/ positive terms

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then both converge or both diverge.

EX Does $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ converge?

EX Does $\sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ converge?