__Review__ Sequence $\{a_n\} = \{a_1, a_2, \dots\}$

__Def.__ $\lim\limits_{n\to\infty} a_n = L$ if $\forall \varepsilon > 0$ there is $N > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > N$.

If so, say $\{a_n\}$ __converges__ to L, otherwise $\{a_n\}$ diverges.

__Useful Thms__

__Monotonic Seq Thm__ Every bounded, monotonic sequence converges

__Thm__ All thms for $\lim\limits_{x\to a} f(x)$ apply (sums, powers, $\cdot$, $\div$, sqz thm)

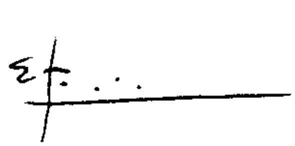
__Ex__ Find limit of $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}\}$ etc.

__A:__ $a_1 = 2^{1/2}$ $a_2 = 2^{3/4}$ $a_3 = 2^{7/8}$ ... $a_n = 2^{\frac{2^n-1}{2^n}}$.

Now $\lim\limits_{n\to\infty} \underbrace{\frac{2^n-1}{2^n}}_{b_n} = 1$ so $\lim\limits_{n\to\infty} 2^{b_n} = 2^{\lim\limits_{n\to\infty} b_n}$ by limit law
$= 2^1 = 2$.

__Thm__ Suppose $\lim\limits_{n\to\infty} |a_n| = 0$. Then $\lim\limits_{n\to\infty} a_n = 0$

"Proof"



if $0 \le |a_n| < \varepsilon$ then $-\varepsilon < a_n < \varepsilon$.

__Ex__ $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$ so $\{1, 1/4, -1/9, 1/16, 1/25, -1/36, 1/49, 1/64, -1/81, \dots\}$
also converges to 0.

Ex $\lim_{n\to\infty} r^n = \begin{cases} 0 & -1<r<1 \\ 1 & r=1 \\ \text{DNE} & \text{otherwise.} \end{cases}$

Ex $\lim_{n\to\infty} \dfrac{\frac{1}{n!}\sin^3(6n)}{n^2}$

Rmk $-1 \le \dfrac{\sin^3(6n)}{n!} \le 1$ so this limit $=0$ by Squeeze thm:

$$\frac{-1}{n^2} \le a_n \le 1/n^2 \text{ and } \lim_{n\to\infty} \frac{-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

SERIES

$a_1+a_2+a_3+\dots+a_n+\dots = \sum_{n=1}^{\infty} a_n$ is called an (infinite) series

Def Given series $\sum a_n$, the n^{th} partial sum is

$$S_n = a_1+a_2+\dots+a_n = \sum_{i=1}^{n} a_i.$$

* Def Given $\sum a_n$ with n^{th} partial sum S_n,

say

$\sum a_n$ is convergent and write $\sum a_n = S$

if

$\lim_{n\to\infty} S_n = S$. If $\lim_{n\to\infty} S_n$ DNE, say

The series diverges.

* Series $\sum a_n$ converges if and only if the sequence of partial sums converges

Ex $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Notice $S_1 = \frac{1}{2}$ $S_2 = \frac{3}{4}$ $S_3 = 7/8 \dots$ $S_n = \frac{2^n - 1}{2^n}$

and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$ Thus $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

Ex Geometric Series $a + ar + ar^2 + \dots$ $(a_1 = a,\ a_n = r a_{n-1})$

$S_n = a + ar + \dots + ar^{n-1}$

$rS_n = ar + ar^2 + \dots + ar^n$ $S_n - rS_n = a - ar^n$

So $S_n = \frac{a - ar^n}{1-r}$ so $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{ar^n}{1-r}$

Conclusion $\sum_{n=1}^{\infty} ar^{n-1}$ converges if and only if $|r| < 1$, so:

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{otherwise.} \end{cases}$$

Ex $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ $a = 1/2$ $r = 1/2$ $\sum = \frac{1/2}{1 - 1/2} = 1$

Ex $6 - 2 + 2/3 - 2/9 + 2/27 \dots = \frac{6}{1 - -1/3} = 6/\frac{4}{3} = 18/4 = 9/2$

Ex $\sum_{n=1}^{\infty} 3^{2n} 5^{1-n} = \frac{9}{5^0} + \frac{3^4}{5^1} + \frac{3^4}{5^2} \cdots$

$a = 9/5 \quad r = \frac{9}{5}$ $\quad \cancel{\frac{9/5}{1-9/5} = \frac{9/5}{-4/5} = -9/4}$

DIVERGES

Ex Repeating decimals are just geometric series!

$$2.1313131\overline{31} = 2.1 + \underbrace{.031 + .00031 + .0000031 + \cdots}_{a = .031 \quad r = 1/100}$$

$$= 2.1 + \frac{.031}{1 - 1/100} = 2.1 + \frac{.031}{99/100} = 2.1 + \frac{31}{990}$$

__Rmk__ Quite unusual to have formula for S_n, geometric series is special, another is so-called __telescoping series__

Ex $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ by partial fractions so

$$\underbrace{\underbrace{\underbrace{1 - \frac{1}{2}}_{s_1} + \frac{1}{2} - \frac{1}{3}}_{s_2} + \frac{1}{3} - \frac{1}{4}}_{s_3} + \frac{1}{4} - \frac{1}{5}$$

so $S_n = 1 - \frac{1}{n+1}$

Thus $\lim_{n \to \infty} S_n = 1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Ex __Harmonic Series__ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (see last class!

Some Theory

Thm If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof Notice that $a_n = S_n - S_{n-1}$. If $\{S_n\}$ converges
Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = S$, so $\lim_{n\to\infty} S_n - S_{n-1} = 0$.

COR Suppose $\lim_{n\to\infty}$ a_n ~~DNE~~ DNE or $\lim_{n\to\infty} a_n \neq 0$. Then $\sum_{n=1}^{\infty} a_n$ diverges

WARNING The condition that $\lim_{n\to\infty} a_n = 0$ is

NECESSARY BUT **NOT SUFFICIENT** for $\sum_{n=1}^{\infty} S_n = 0$.

Think We can't possible add infinitely many things and get a value unless those things are $\to 0$

Ex $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$

COR DOES NOT APPLY!!

Ex $\sum_{n=1}^{\infty} \frac{6n^3}{n^3+1}$ DIVERGES since $\lim_{n\to\infty} \frac{6n^3}{n^3+1} = 6 \neq 0$.

Thm Suppose $\sum a_n = S$ and $\sum b_n = T$ converge. Then

1. $\sum c a_n = cS$ 2. $\sum (a_n \pm b_n) = S \pm T$

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<u>Problems</u>

1. $1 + .4 + .16 + .064 + \ldots$

2. $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{3})^{2n}}$

3. $\sum_{n=1}^{\infty} \frac{e^n}{n^5}$

4. $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ use Telescoping