Lecture 17

Sequences & Series

Motivation

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \ ? \]

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \ ? \]

Answer:

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \]

\[ > \frac{1}{16} > \frac{1}{16} > \frac{1}{16} > \frac{1}{16} \]

So \[ 1 + \frac{1}{2} + \frac{1}{4} + \ldots \] diverges

Claim:

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \]

Goal: Make this precise by considering sequence of partial sums

\[ S_1 = \frac{1}{2}, \ S_2 = \frac{3}{4}, \ S_3 = \frac{7}{8}, \ \ldots \]

SEQUENCES

A sequence is an infinite list of numbers

\[ \{ a_1, a_2, a_3, \ldots, a_n, \ldots \} = \{ a_n \} = \{ a_n \}_{n=1}^{\infty} \]

Ex. \[ 1, 2, 4, 8, 16, \ldots \]

2. \[ 3, 3.1, 3.14, 3.141, \ldots \]

How to give a sequence?

1. Formula for \( a_n \)

Ex. \[ \{ 2^n+1 \}_{n=1}^{\infty} = \{ 3, 5, 7, 9, 11, \ldots \} \]

Ex. \[ \{ (-1)^n 2^{n+1} \}_{n=3}^{\infty} = \{ -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, \ldots \} \]
1. Rule for \( n^{th} \) term
   \[ a_n = \frac{10}{9} \text{ to } n \text{ digits} \quad \{3.31, 3.14, 3.141, \ldots \} \]

2. Recursively:
   - Give beginning of sequence & rule for determining
     \( a_n \) in terms of \( a_{n-1}, a_{n-2}, \ldots \) etc.
   - Famous Example: Fibonacci Sequence
     \[ f_1 = f_2 = 1 \]
     \[ f_n = f_{n-1} + f_{n-2} \quad n \geq 3 \]
     \[ \{1, 1, 2, 3, 5, 8, 13, 21, \ldots \} \text{ fn called } n^{th} \text{ Fibonacci} \]

Think of sequence as function with \( n \) on x-axis

3. Example:
   \[ \{1+\frac{1}{n}\} = \{a_n\} \]

   \[ \lim_{n \to \infty} a_n = 1 \]

- Def: Informal
  \[ \lim_{n \to \infty} a_n = L \text{ if however close we choose } (\varepsilon > 0) \]
  the sequence eventually gets that close and stays that close.

- Def: Formal
  A sequence \( \{a_n\} \) has limit \( L \), denoted \( \lim_{n \to \infty} a_n = L \) if
  for every \( \varepsilon > 0 \) there is an \( N > 0 \) such that \( |a_n - L| < \varepsilon \) whenever
  \( n > N \). It say \( \{a_n\} \) converges, else it diverges.
Review almost identical let for \( \lim_{x \to 0} f(x) = L \).

Then suppose \( \lim_{x \to 0} f(x) = L \). Let \( a_n = f(n) \) be sequence. Then \( \lim_{n \to \infty} a_n = L \).

\[
\text{Picture: } \begin{array}{c}
\bullet \\
L \\
\text{as } r \to \infty \\
\end{array}
\]

Ex. Let \( a_n = \cos^{-1}(n) = (\cos^{-1} 1 \cdot \cos^{-1} 2 \ldots) \). The \( \lim a_n = \frac{\pi}{2} \).

Limit laws. We can add sequence and multiply by constants and multiply and divide.

Ex. \( \{a_n\} = \{1, 1/2, 1/3, 1/4, 1/5, \ldots\} \)
(\( \{b_n\} = \{1, 2, 3, 4, 5, \ldots\} \))
(\( \{c_n\} = \{2, 4, 6, 8, 10, \ldots\} \))
(\( \{d_n\} = \{2, 2^2, 2^3, 2^4, \ldots\} \))

Then suppose \( \{a_n\}, \{b_n\} \) converge. Then

1. \( \lim_{n \to \infty} (a_n \pm b_n) = (\lim_{n \to \infty} a_n) \pm (\lim_{n \to \infty} b_n) \)
2. \( \lim_{n \to \infty} (a_n \cdot b_n) = (\lim_{n \to \infty} a_n) \cdot (\lim_{n \to \infty} b_n) \)
3. \( \lim_{n \to \infty} (a_n \cdot b_n) = (\lim_{n \to \infty} a_n) \cdot (\lim_{n \to \infty} b_n) \)
4. \( \lim_{n \to \infty} c \cdot a_n = c \lim_{n \to \infty} a_n \)
5. \( \lim_{n \to \infty} a_n = a \)
\[
\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}.
\]

Some more useful limit theorems:

**Theorem 1.** If \( \lim_{n \to \infty} f(x) = L \) then \( \lim_{n \to \infty} \{f(n)\} = L \) where \( f(n) = f(an) \).

2. (Squeeze Theorem) If \( a_n \leq b_n \leq c_n \) for all \( n \geq N \), then if \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then \( \lim_{n \to \infty} b_n = L \).

3. Suppose \( \lim_{n \to \infty} |a_n| = 0 \). Then \( \lim_{n \to \infty} a_n = 0 \).

**Example**

1. Does \( \left\{ \frac{n^2}{e^n} \right\}_{n=0}^{\infty} \) converge/diverge and what is the limit?

   \[ f(x) = \frac{x^2}{e^x} \text{ and this sequence is } f(n). \]

   \[
   \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.
   \]

   Thus sequence converges to 0.
Def: A sequence \( \{a_n\} \) is increasing if \( a_n < a_{n+1} \) for all \( n \).
A sequence \( \{a_n\} \) is decreasing if \( a_n > a_{n+1} \) for all \( n \).
A sequence is monotonic if it is increasing or decreasing.
A sequence is bounded above if \( \exists \) a number \( M \) s.t. \( a_n \leq M \) \( \forall n \).
A sequence is bounded below if \( \exists \) a number \( m \) s.t. \( m \leq a_n \) \( \forall n \).
A sequence is bounded if it is bounded above and below.

The Every bounded, monotonic sequence converges.

Rmk: increasing + bounded above \( \Rightarrow \) bounded
     decreasing + bounded below \( \Rightarrow \) bounded.