Lecture 17

Sequences & Series

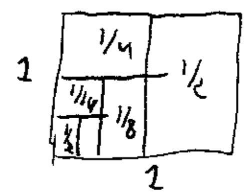
Motivation $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots = ?$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = ?$

Answer 1: $1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>1/2}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>1/2}+\underbrace{\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}}_{>1/2}+\underbrace{\frac{1}{17}+\dots+\frac{1}{32}}_{>1/2}\dots$

So $1+\frac{1}{2}+\frac{1}{3}+\dots = \sum_{i=1}^{\infty}\frac{1}{i}$ diverges

Answer 2



Claim: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = \sum_{i=1}^{\infty}\frac{1}{2^i} = 1.$

Goal Make this precise by considering <u>sequence of partial sums</u>

$s_1 = \frac{1}{2},\ s_2 = \frac{1}{2}+\frac{1}{4},\ s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, etc.

SEQUENCES

Def A <u>sequence</u> is an infinite list of #'s

$$\{a_1, a_2, a_3, \dots, a_n, \dots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

Ex 1. $\{1, 2, 4, 8, 16, \dots\}$

2. $\{3, 3.1, 3.14, 3.141, \dots\}$

How to give a sequence?

1. Formula for a_n

Ex $\{2n+1\}_{n=1}^{\infty} = \{3, 5, 7, 9, 11, \dots\}$

Ex $\{(-1)^n/2^n\}_{n=3}^{\infty} = \{-\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, \dots\}$

1.5 "Pattern" $\{1,0,1,0,0,1,0,0,0, \text{etc.}\}$

2. Rule for n^{th} term
   Ex $a_n = \pi$ to n digits $\{3, 3.1, 3.14, 3.141, \ldots\}$

3. Recursively:
   - Give beginning of sequence + rule for determining a_n in terms of $a_{n-1}, a_{n-2}, \ldots$ etc.

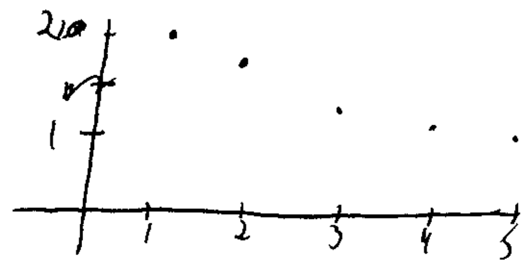
Famous Example: <u>Fibonacci Sequence</u>

$$f_1 = f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ f_n called <u>n^{th} Fibonacci #</u>.

Think of sequence as function with n on x-axis

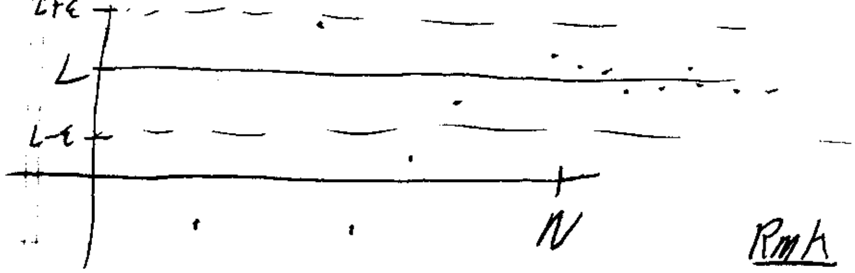
<u>Ex</u> $\{1 + 1/n\} = \{a_n\}$



Want $\lim\limits_{n \to \infty} a_n = 1$.

<u>Informal Def</u> $\lim\limits_{n \to \infty} a_n = L$ if however close we choose $(\varepsilon > 0)$ the sequence <u>eventually</u> gets that close and <u>stays</u> that close.

<u>Formal Def</u> A sequence $\{a_n\}$ has limit L, denoted $\lim\limits_{n \to \infty} a_n = L$, if for every $\varepsilon > 0$ there is an $N > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > N$. If so, say $\{a_n\}$ <u>converges</u>. Else it <u>diverges</u>.



Rmk Smaller ε may require larger N!

Ex

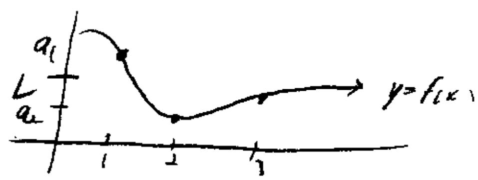
1. $\{1, -1/2, 1/3, -1/4, 1/5, \ldots\}$ $\quad \lim_{n\to\infty} a_n = 0$

2. $\{.1, 1, .01, 1, .001, 1, .0001, 1, \ldots\}$ diverges.

Review almost identical def for $\lim_{x\to\infty} f(x) = L$.

Thm Suppose $\lim_{x\to\infty} f(x) = L$. Let $a_n = f(n)$ be sequence. Then $\lim_{n\to\infty} a_n = L$.

Picture:



Ex Let $\{a_n\} \tan^{-1}(n) = \{\tan^{-1} 1, \tan^{-1} 2, \ldots\}$ the $\lim_{n\to\infty} a_n = \pi/2$

Limit laws We can add sequence and multiply by constants:
and multiply and divide

Ex $\{a_n\} = \{1, 1/2, 1/3, 1/4, 1/5\}$ $\quad \{b_n\} = \{1, 2, 3, 4, \ldots\}$
$\{2a_n\} = \{2, 1, 2/3, 2/4, 2/5 \ldots\}$ $\quad \{a_n + b_n\} = \{2, 2\tfrac{1}{2}, 3\tfrac{1}{3}, 4\tfrac{1}{4}, \ldots\}$

Thm Suppose $\{a_n\}, \{b_n\}$ converge. Then

1. $\lim_{n\to\infty} \{a_n \pm b_n\} = \lim_{n\to\infty}\{a_n\} + \lim_{n\to\infty}\{b_n\}$
2. $\lim_{n\to\infty} \{c a_n\} = c \lim_{n\to\infty}\{a_n\}$
3. $\lim_{n\to\infty} \{a_n b_n\} = \lim_{n\to\infty}\{a_n\} \lim_{n\to\infty}\{b_n\}$
4. $\lim_{n\to\infty} \{c\} = c$

3 $\lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = \lim_{n\to\infty}\{a_n\} \Big/ \lim_{n\to\infty}\{b_n\}$.

Some more useful limit thms:

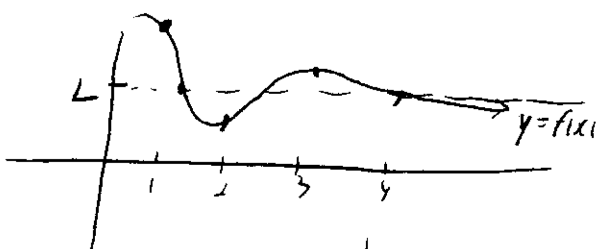
Thm 1. If $\lim_{n\to\infty} f(x) = L$ then $\lim_{n\to\infty}\{f_n\} = L$ where $f_n = f(n)$.

2. (Squeeze Thm) If $a_n \le b_n \le c_n$ for all $n \ge N$, ~~th~~ and if $\lim_{n\to\infty}\{a_n\} = \lim_{n\to\infty}\{c_n\} = L$, then $\lim_{n\to\infty}\{b_n\} = L$.

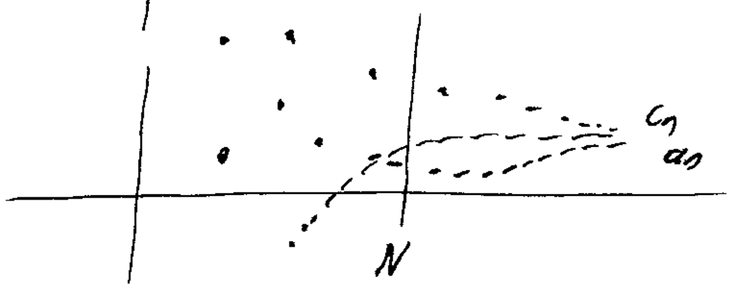
3. Suppose $\lim_{n\to\infty} |a_n| = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Pictures

1.



2.



"eventually" b_n is squeezed.

Ex

1. Does $\left\{\frac{n^2}{e^n}\right\}_{n=0}^{\infty}$ converge / diverge and what is limit?

$f(x) = \frac{x^2}{e^x}$ and this sequence is $f(n)$.

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

Thus sequence converges to 0.

<u>Def</u> A sequence $\{a_n\}$ is <u>increasing</u> if $a_n < a_{n+1}$ for all n
A sequence $\{a_n\}$ is <u>decreasing</u> if $a_n > a_{n+1}$ for all n
A sequence is <u>monotonic</u> if it is increasing or decreasing.
A sequence is <u>bounded above</u> if $\exists$ a number M s.t

$$a_n \leq M \quad \forall n.$$

A sequence is <u>bounded below</u> if $\exists$ m such that

$$m \leq a_n \quad \forall n.$$

A sequence is <u>bounded</u> if it is bounded above and below.

<u>Thm</u> Every bounded, monotonic sequence converges.

<u>Rmk</u> increasing + bounded above $\Rightarrow$ bounded
decreasing + bounded below $\Rightarrow$ bounded