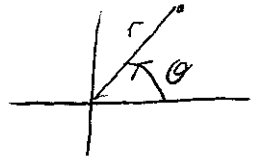
Review

Polar coordinates



$x = r\cos\theta \quad y = r\sin\theta$

$r = \sqrt{x^2+y^2} \quad \tan\theta = y/x$

Warning: A single point has infinitely many ways to be represented in polar coordinates.

Ex $(r, \theta) = (-r, \theta+\pi)$

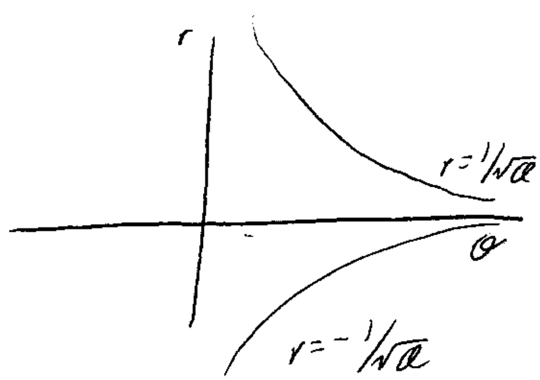
$(r,\theta) = (r, \theta \pm 2\pi n)$ $n = 1, 2, \ldots$ $\qquad (0,\theta) = (0,0)$ all same

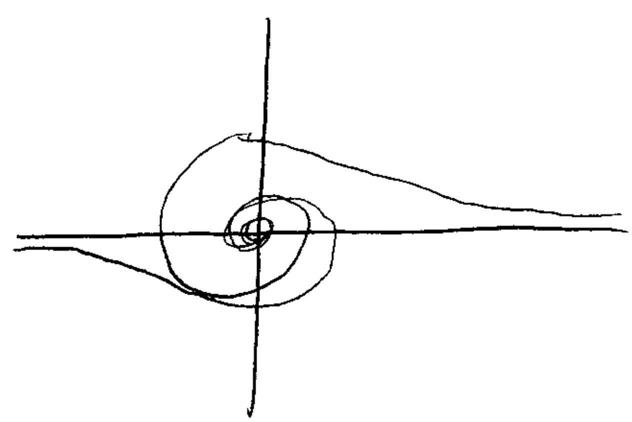
Rmk $(0,0) \cup \{(r,\theta) \mid r>0 \text{ and } 0 \le \theta < 2\pi\}$ gives each point exactly once.

Polar Equations

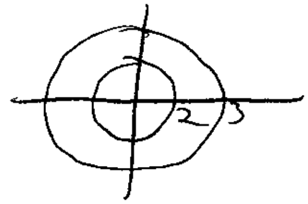
Plot every point that has at least one polar representation satisfying the equation

Ex Sketch $r^2\theta = 1$, so $r = \pm 1/\sqrt{\theta}$





Ex $(r-3)(r+2)=0$



Ex Lots of other interesting curves, think Spirograph.

Calculus of Polar Curves

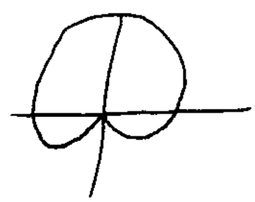
Suppose $r=f(\theta)$. Trick: Think of θ as a parameter:

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then: $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Do not memorize! Same formula from $(x(t), y(t))$.

Ex Cardioid $r = 1+\sin\theta$



- Find Hor & Vert tangents
- Slopes of tang lines at origin?

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

$$= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

Rmk $0 \le \theta \le 2\pi$ sweeps whole curve.

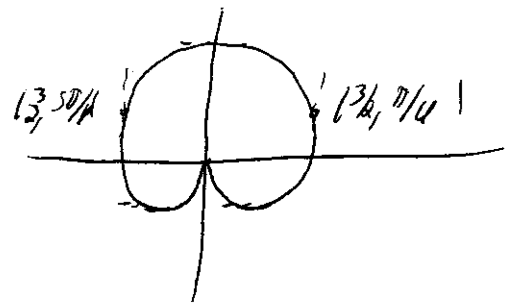
Hor tang $\cos\theta = 0$ or $\sin\theta = -1/2$

$\theta = \pi/2, 3\pi/2, 5\pi/2, \ldots$ or $\theta = 7\pi/6, 11\pi/6, \ldots$

VERT TANG $1 + \sin\theta = 0$ or $1 - 2\sin\theta = 0$

$\theta = 3\pi/2$ $\theta = \pi/6, 5\pi/6$

Warning At $\theta = 3\pi/2$, $\frac{dy}{dx} \to \frac{0}{0}$, use L'Hospital to get ∞

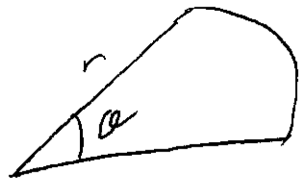


so vertical tang at $(0,0)$

Problem Show $r = a\cos\theta$ and $r = a\sin\theta$ intersect at right angles.

Areas and Arc length

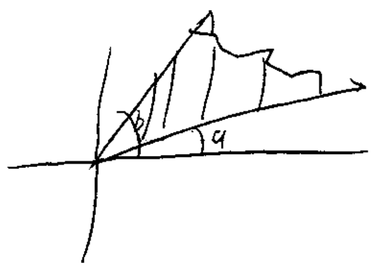
Formula



Area $= \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$

So $A = \frac{1}{2} r^2 \theta$ is area of wedge.

Problem Given $r = f(\theta)$ for $a \le \theta \le b$, $0 < b-a \le 2\pi$ and $f(\theta) \ge 0$.



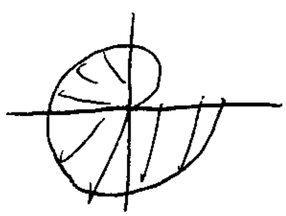
what is area?

Divide into wedges: $\theta = \Delta\theta$ ~~A~~ $r = f(\theta)$

~~Thm~~ The area above is $A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$

$$= \int_a^b \frac{1}{2} r^2 \, d\theta$$

Ex $r = \sqrt{\theta}$

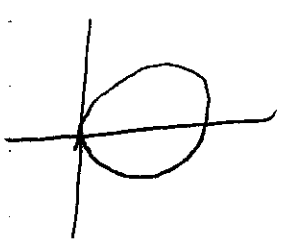


Find area.

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \theta \, d\theta = \frac{\theta^2}{4} \Big|_0^{2\pi}$$

$$= \pi^2$$

Ex Find area of one leaf of 4 leave rose $r = \cos 2\theta$



$-\pi/4 \le \theta \le \pi/4$ gives one leaf.

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{4} (1 + \cos 4\theta) \, d\theta$$

$$= \theta/4 + \frac{1}{8} \sin 4\theta \Big|_{-\pi/4}^{\pi/4} = \pi/8$$

Arc length $r = f(\theta)$ $a \le \theta \le b$

As before $x = r\cos\theta$ $y = r\sin\theta$

Check $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$

Thus

Thm Length of curve $r = f(\theta)$ $a \le \theta \le b$ is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

In[1]:=

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2 + 2
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Out[1]= 4

In[15]:=

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PolarPlot[{1 / (t^(1 / 2)), -1 / (t^(1 / 2))}, {t, 1, 100}]
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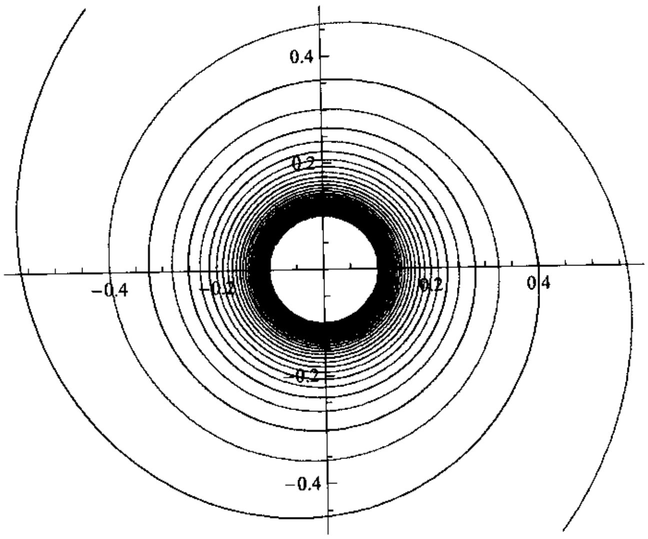
Out[15]=



In[16]:=

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PolarPlot[2 Sin[t] + 3 Sin[9 t], {t, 1, 8}]
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Out[16]=