Polar coordinates

$$\sqrt{x^2+y^2} = r$$

$$\tan \Theta = \frac{y}{x}$$

$$x = r \cos \Theta$$

$$y = r \sin \Theta$$

$$r = \sqrt{x^2+y^2}$$

Warning: A single point has infinitely many ways to be represented in polar coordinates.

Ex. \((r, \Theta) = (-r, \Theta + \pi)\)

\((r, \Theta) = (r, \Theta + 2\pi n) \quad n = \text{integers} \quad (0, \Theta) = (0, \Theta) \text{ all same}\)

Remark \((0, \Theta) \cup \{(r, \Theta) \mid r > 0 \text{ and } 0 \leq \Theta < 2\pi\}\)

gives each point exactly once.

Polar Equations

Plot every point that has at least one polar representation satisfying the equation.

Ex. Sketch \(r^2 \Theta = 1\), so \(r = \pm \frac{1}{\sqrt{\Theta}}\)

\(\text{Diagram of a spiral curve with the equation } r^2 \Theta = 1\)
Ex. \((r-3)(r+3) = 0\)

Ex. Lots of other interesting curves, think Spirograph.

Calculus of Polar Curves

Suppose \(r = f(\theta)\). Trick: Think of \(\theta\) as a parameter:

\[
x = r \cos \theta = f(\theta) \cos \theta \\
y = r \sin \theta = f(\theta) \sin \theta.
\]

Then:

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}
\]

Do not memorize! Some formula from (x,y) to (\(x',y'\)).

Ex. Cardioid \(r = 1 + \sin \theta\)

Find the vet tangents. Slopes of tan lines at origin?

\[
\frac{dy}{dx} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos^2 \theta - (1 + \sin \theta) \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{\cos^2 \theta - \sin^2 \theta - \sin \theta}
\]

\[
= \frac{\cos \theta (1 + 2 \sin \theta)}{1 - 2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{(1 - \sin \theta)(1 - 2 \sin \theta)}
\]
Rank $0 \leq \cos \theta \leq 1$ or $\sin \theta = -\frac{1}{2}$

$\theta = \pm \frac{1}{6}, \pm \frac{3}{6}, \pm \frac{5}{6}, \ldots$ or $\theta = \frac{7}{6}, \frac{11}{6}, \ldots$

Vertical:
$1 + \sin \theta = 0$ or $1 - 2 \sin \theta = 0$

$\theta = \frac{3}{6}$

$\theta = \frac{7}{6}, \frac{5}{6}$

Warning: At $\theta = \frac{3}{6}$, $\frac{dy}{dx} \to \frac{0}{0}$, use L'Hôpital's rule to get $\infty$.

Problem: Show $r = a \cos \theta$ and $r = a \sin \theta$ intersect at right angles.
Areas and Arc length

Formula

\[ \text{Area} = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta \]

So \( A = \frac{1}{2} r^2 \theta \) is area of wedge.

Problem (Given \( r = f(\theta) \) for \( a \leq \theta \leq b \), \( 0 < b - a \leq 2\pi \) and \( f(\theta) > 0 \)).

Divide into wedges: \( \theta = \Delta \theta \) \( A = r = f(\theta) \)

Then the area above is

\[ A = \frac{1}{2} \int_a^b f(\theta)^2 \, d\theta \]

\[ = \frac{1}{2} \int_a^b \theta \, d\theta \]

Ed: \( r = \sqrt{\theta} \)

Find area.

\[ A = \frac{1}{2} \int_0^{2\pi} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} \theta \, d\theta = \frac{\theta^2}{4} \bigg|_0^{2\pi} \]

\[ = \pi^2 \]

Ed: Find area of one leaf of 4-leaf rose \( r = \cos 2\theta \)

\[ \frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4} \] gives one leaf.

\[ A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) \, d\theta \]

\[ = \frac{\pi}{4} + \frac{1}{2} \sin 4\theta \bigg|_{-\pi/4}^{\pi/4} = \frac{\pi}{8} \]
Arc length $r = f(\theta)$ $a \leq \theta \leq b$

As before $x = r \cos \theta$ $y = r \sin \theta$

Check $\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 = \left( \frac{dr}{d\theta} \right)^2 + r^2$

Thus

The length of curve $r = f(\theta)$ $a \leq \theta \leq b$ is

$$L = \int_a^b \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta$$
ln[1] := 2 + 2

Out[1] := 4

ln[15] := PolarPlot[{1/(t^(1/2)), -1/(t^(1/2))}, {t, 1, 100}]

Out[15] :=

ln[16] := PolarPlot[2 Sin[t] + 3 Sin[9 t], {t, 1, 8}]

Out[16] :=