Lecture 15

Review

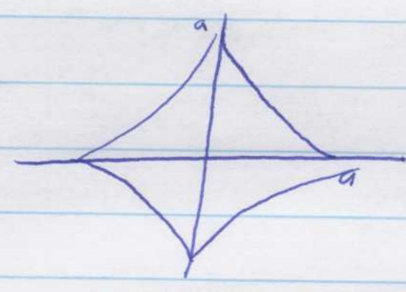
Parameterized curve $(x(t), y(t))$ $a \le t \le b$.

- Velocity vector $(x'(t), y'(t))$
- Slope of tangent is $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = y'(t)/x'(t)$
- Arc Length (Distance travelled) $= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$
  where speed $= \sqrt{x'(t)^2 + y'(t)^2}$

Areas
- Recall if $y = F(x) \ge 0$, $a \le x \le b$ then Area under curve is $\int_a^b F(x)\,dx$
- If curve is $(x(t), y(t))$ then

$$\text{Area} = \int_a^b y\,dx = \int_a^b y(t)\,x'(t)\,dt$$

Ex Find area enclosed by astroid $x = a\cos^3\theta$ $y = a\sin^3\theta$



Solution $\text{Area} = 4 \cdot \int_{\pi/2}^{0} a\sin^3\theta \cdot 3a\cos^2\theta(-\sin\theta\, d\theta)$

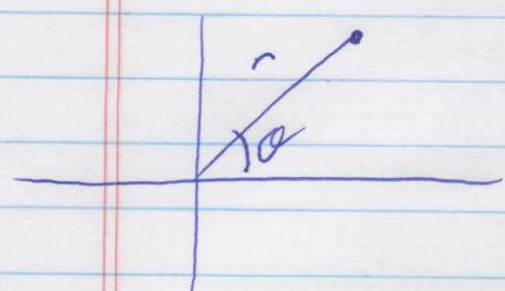
$$= -12a^2 \int_0 \sin^4\theta \cos^2\theta\, d\theta = \frac{12a^2\pi}{32}$$

Surface Area Suppose $x=f(t)$ $y=g(t)$ $a\le t\le b$ about x axis, $g(t)\ge 0$.
Then
$$SA=2\pi\int_a^b y\,ds = 2\pi\int_a^b g(t)\sqrt{f'(t)^2+g'(t)^2}\,dt$$

Rmk Typically parametrized curves meander all over, so area & SA formulas here not so common

Problem $x=\sin^2 t$ $y=\cos^2 t$ $0\le t\le 3\pi$. Find distance travelled by particle, compare w/ length of curve.

Polar Coordinates



For a point in (x,y) plane, let

$r=\sqrt{x^2+y^2}$ be distance from origin,

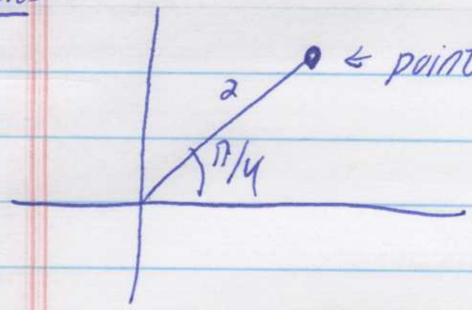
$\theta = \angle$ from positive x axis ccwise

$\tan\theta = y/x$

Rmk For points $\neq (0,0)$ there is a unique coordinate $r>0$ and $0\le\theta<2\pi$

Rmk If $r>0$ then $(-r,\theta)$ lies on same line as (r,θ) but opp direction

Examples



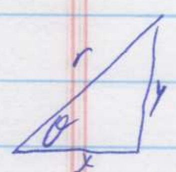
← point is $(2, \pi/4)$ in polar.

A.K.A. $(-2, 5\pi/4)$

AKA $(2, 9\pi/4)$ etc..

Rmk (r, θ) is obviously same as $(r, \theta + 2n\pi)$ $n = \ldots -2, -1, 0, 1, 2, \ldots$
and same as $(-r, \theta + \pi + 2n\pi)$ " "

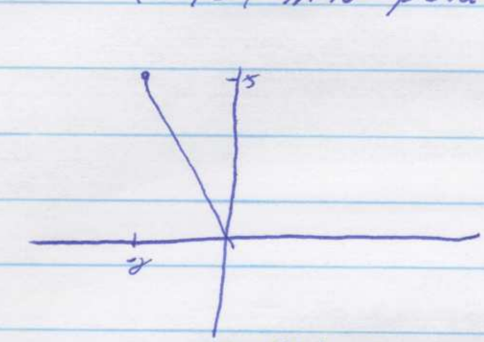
Converting Back and Forth



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2+y^2} \qquad \tan\theta = y/x$$

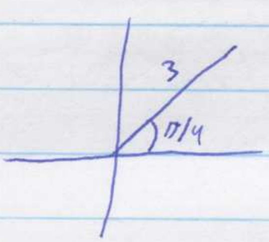
Ex Put $(-2, 5)$ into polar coord



$$r = \sqrt{29}$$

$$\theta = \tan^{-1}\left(\frac{-5}{2}\right)$$

Ex Convert $(3, \pi/4)$ from polar to Cartesian



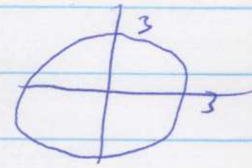
$$x = 3\cos(\pi/4) = 3\sqrt{2}/2$$

$$y = 3\sin(\pi/4) = 3\sqrt{2}/2$$

Sketching Polar Equations

Rmk Some curves (and regions) are simpler when described in polar coordinates

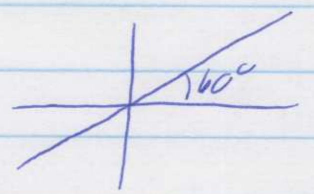
Ex Sketch $r=3$



$$\sqrt{x^2+y^2}=3$$

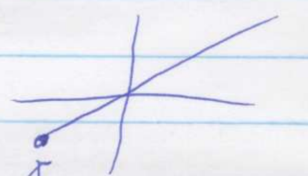
Ex Region $1 \le r \le 2$

Ex $\theta = \pi/3$



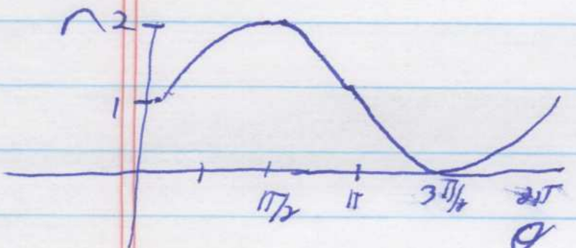
Def The graph of a polar equation $F(r,\theta)=0$ is <u>all points</u> that have <u>at least one</u> polar representation (r,θ) which satisfies equation.

Ex $\theta = \pi/4$

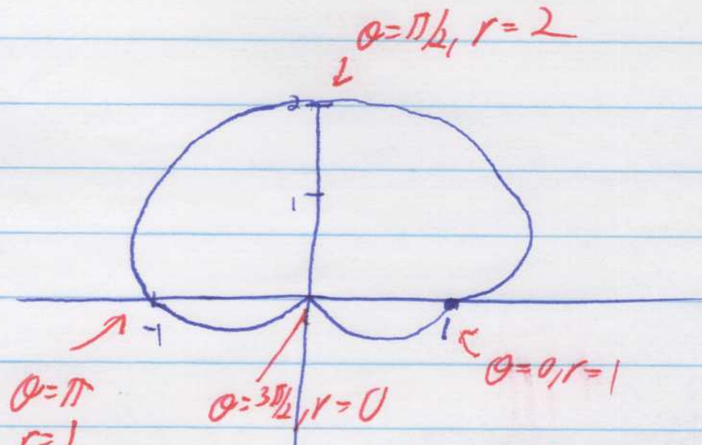


$(1, 5\pi/4) = (-1, \pi/4)$ so point is on curve!

Ex Sketch $r = 1 + \sin\theta$

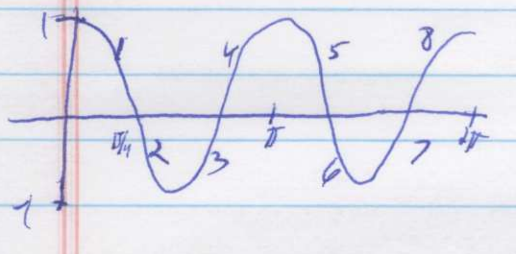


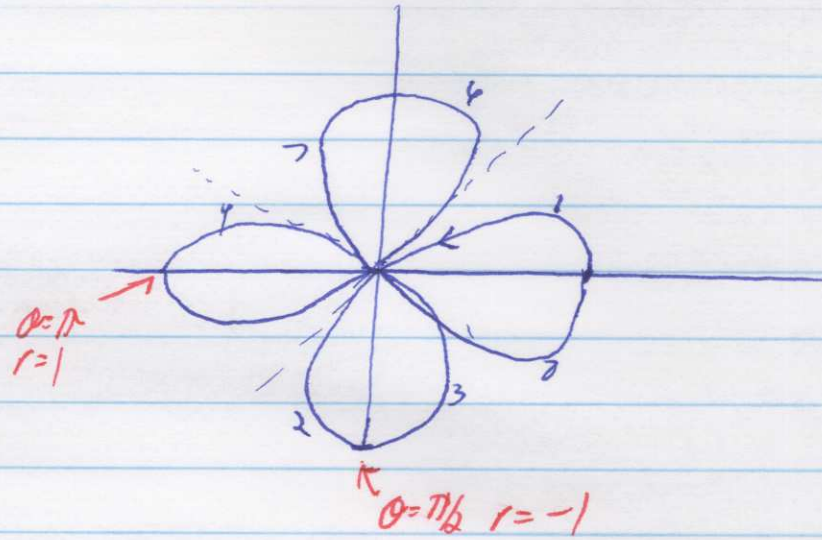
$\Longrightarrow$



cardioid

Ex $r = \cos(2\theta) \quad 0 \le \theta \le 2\pi$





four-leave rose.

Tips For Sketching

1. Plot curve in r, θ plane to help

2. Symmetries : For example $r = \cos(2\theta)$ but $\cos(2\theta) = \cos(-2\theta)$ so graph must be symmetric about origin.

Tangents to Polar Curves $r = f(\theta)$

Suppose $r = f(\theta)$ is a polar curve Then:

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Don't memorize this!

Problems

1. $r = \cos(\theta/3)$ Find tangent line at $\theta = \pi/4$.

   $y = \cos(\theta/3)\sin\theta$

   $x = \cos(\theta/3)\cos\theta$ etc.

2. Let $r = e^{\theta}$, find points w/ vertical or horiz tangents. Sketch curve.

3. $x^2 + y^2 = 5x$, put in polar equation

4. Sketch $r = \ln\theta$ $\theta \geq 1$

5.