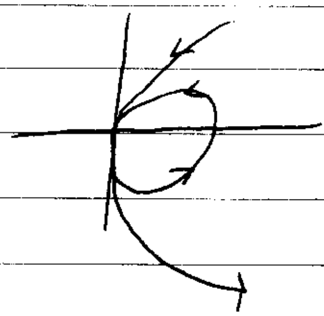
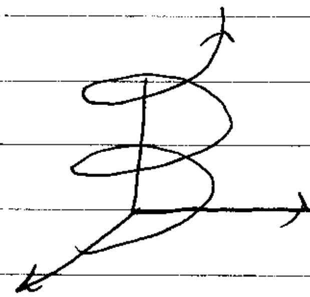
Lecture 14

Parametric Equations

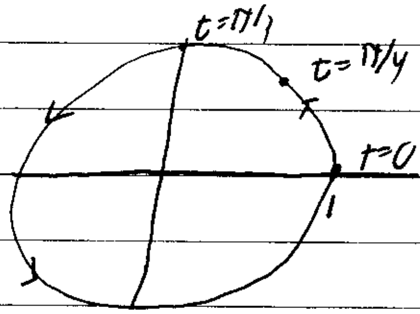
Problem Many curves in space are not graph of a function $y = f(x)$ or $x = g(y)$.

Ex 

Ex 

Solution Give each coordinate $x, y, z, \ldots$ as a function of a variable t, called a parameter.

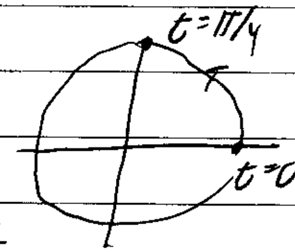
Ex $x = \sin t$, $y = \cos t$



- Called a parametrized curve

- Observe that every point $(\sin t, \cos t)$ lies on circle $x^2+y^2=1$, however the parameterization carries more information

E.g. $(x, y) = (\sin 2t, \cos 2t)$



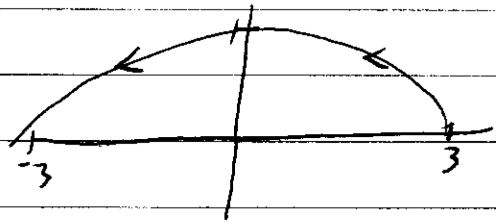
goes around twice as fast

E.g. $(x,y) = (\cos t, \sin t)$ goes around clockwise.

Parametric equations tell __when__ not just where.

Ex $(x,y) = (3\cos t, \sin t)$ $0 \le t \le \pi$

Notice $\frac{x^2}{9} + \frac{y^2}{1} = 1$



top half of ellipse.

__Sketching Parametrized curves__

Method 1. Plot lots of points! (e.g. on a computer)

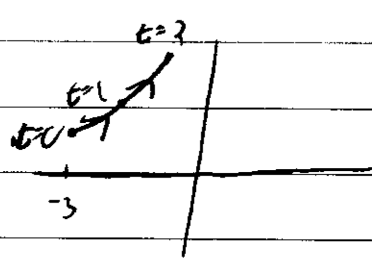
Ex $(x,y,z) = (e^3\cos t, \ln t, e^t + 5t)$ $0 < t < \infty$

__Method 2__ Discover some relation satisfied among $x, y, \ldots$ to help

Ex $(x,y) = (\cos t, \sin t)$ used that $x^2 + y^2 = 1$

Ex $(x,y) = (t-3, t^2+1)$ $0 \le t \le 2$

Notice that $x+3 = t$ so $y = (x+3)^2 + 1$



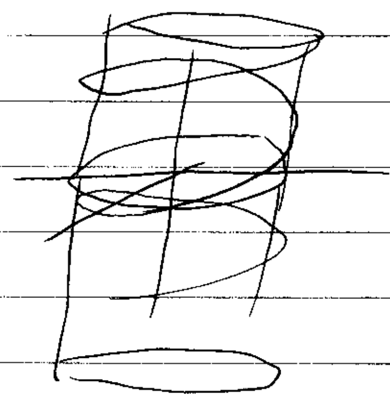
small portion of parabola

Ex $(x,y) = (h + r\cos t, y + r\sin t)$ $\quad 0 \le t \le 2\pi$

Notice that $(x-h)^2 + (y-k)^2 = r^2$

This parametrizes circle, radius r, center (h,k), once around clockwise.

Ex $(x,y,z) = (\cos t, \sin t, t)$ Notice $x^2+y^2=1$, cylinder

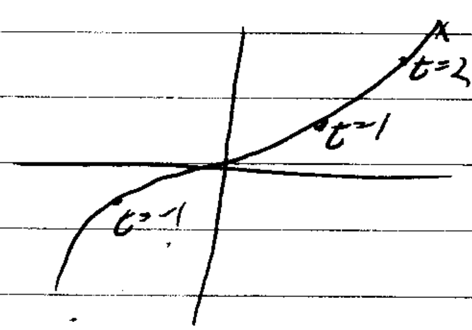


curve stays on cylinder as $z=t$ moves up and down

Ex Graph of $y=f(x)$ easy to parametrize as

$(t, f(t))$

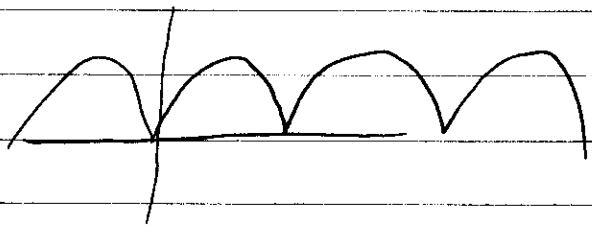
Ex $(x,y) = (t, t^3)$ gives $y = x^3$



Rmk $(x,y) = (t^3, t^9)$ also works, different parametrization.

<u>Famous Examples</u>

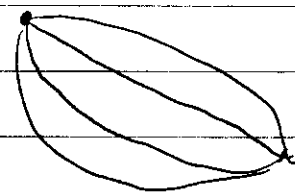
Cycloid: $(x,y) = \left(r(t - \sin t), r(1-\cos t)\right) \quad -\infty < t < \infty$



curve fly on circle of radius r traces as it rolls along.

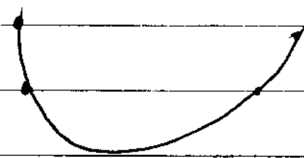
Solution to:

~~Brachistochrone problem~~



fastest slide to bottom.

Tautochrone problem



same time to reach bottom from any point!

<u>Ex</u> Sketch curve w/ arrows, Find Cartesian Equations if possible

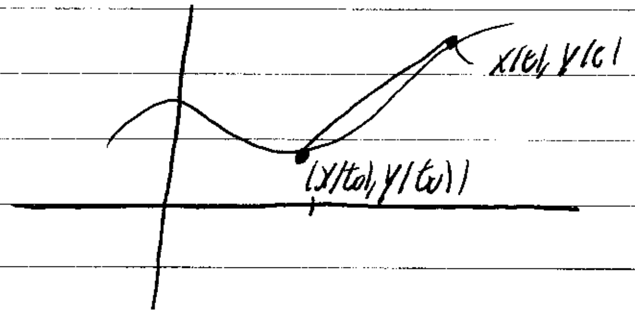
1 $x = e^{2t} \quad y = t + 1$

2 $x = t^2 - 2 \quad y = 5 - 2t \quad -3 \le t \le 4$

3 $x = \sqrt{t} \quad y = 1 - t$

Calculus

Let $(x(t), y(t))$ be a parametrized curve. What is a tangent vector?



$$\lim_{t \to 0} \left(\frac{x(t_0+h) - x(t_0)}{h}, \frac{y(t_0+h) - y(t_0)}{h} \right)$$

$$= (x'(t), y'(t))$$

Def $(x'(t), y'(t))$ is called the velocity vector or tangent vector.

1. It points tangent to the curve, in particular the slope of the tangent line is

$$\frac{y'(t)}{x'(t)}$$

2. It's length $\sqrt{x'(t)^2 + y'(t)^2}$ measures speed.

Ex 1. $x = t^2 \quad y = t^3 - 3t$

a. Find tangent line when $t = 2$

b. Any horizontal or vertical tangents?

c. Sketch

2. Circle case

<u>Arc length</u> • As before, but line segment has length

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \quad \text{thus}$$

<u>Thm</u> Given a curve $x=f(t), y=g(t)$ $\alpha \le t \le \beta$, then Distance travelled is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

• Warning: Only arc length if curve traversed once:

e.g. $(\cos t, \sin t)$ $0 \le t \le 2\pi$, integral will give <u>twice</u> circumference

• Suppose $y=f(x)$. Parametrize as $(t, f(t))$

Then $A.L. = \int_a^b \sqrt{1+f'(t)^2}\, dt$

our old formula.