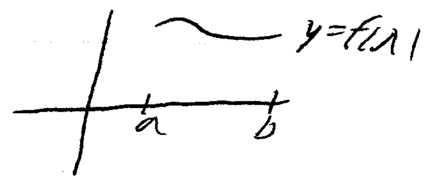
Review

Curve $y = f(x) \geq 0$ for $a \leq x \leq b$



Area under curve: $\int_a^b f(x)\,dx$

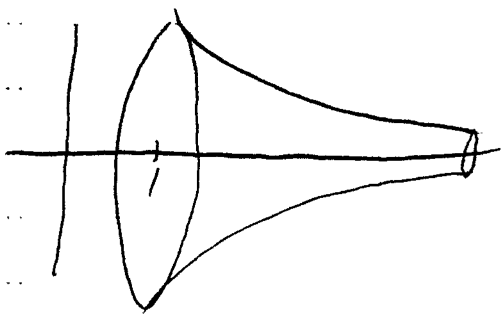
Volume of revolution around x axis: $\int_a^b \pi f(x)^2\,dx$

Length of curve $= \int_a^b \sqrt{1+f'(x)^2}\,dx$

Surface Area of Revolution $SA = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$

Warning: This list is not complete, we can also do more complicated volumes (washers, shells), areas between curves and rotations around other lines!

Dave's Favorite Example Gabriel's horn: $y = 1/x$ $x \geq 1$ rotate around x axis



Volume $= \int_1^\infty \pi \cdot \left(\frac{1}{x}\right)^2 dx = \int_1^\infty \frac{\pi}{x^2}\,dx = -\frac{\pi}{x}\Big|_1^\infty = 0 - -\pi = \pi$

S.A. $= \int_1^\infty 2\pi \cdot \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$

$$= 2\pi \int_1^\infty \frac{\sqrt{1+\frac{1}{x^4}}}{x}\,dx = 2\pi \int_1^\infty \frac{\sqrt{\frac{x^4+1}{x^4}}}{x}\,dx$$

$$= 2\pi \int_1^\infty \frac{\sqrt{\frac{x^4+1}{x^2}}}{x^2}\,dx \qquad u=\frac{1}{x} \quad du = -\frac{1}{x^2}\,dx$$

$$= -2\pi \int_1^0 \sqrt{u^2+1/u^2}\,du = 2\pi \int_0^1 \sqrt{u^2+1/u^2}\,du$$

$$\geq 2\pi \int_0^1 \sqrt{u^2-2+1/u^2}\,du = 2\pi \int_0^1 \sqrt{(u-1/u)^2}\,du$$

$$= 2\pi \int_0^1 u - 1/u\,du = 2\pi\left(\frac{u^2}{2} - \ln|u|\right)\Big|_0^1$$

DIVERGES!

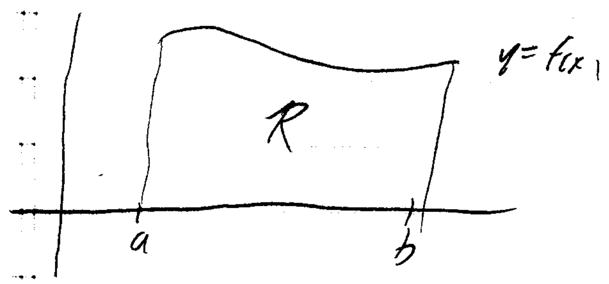
<u>Rmk</u> Horn has finite volume (can "fill" with paint) but infinite surface area (can't be painted)

<u>Thm</u> Suppose $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

1. If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ converges
2. If $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty f(x)\,dx$ diverges

We used similar comparison above for improper integral $\int_0$.

Selected applications from 8.3–8.5



Let A = area R ($= \int_a^b f(x)\,dx$)

Goal Suppose R has uniform density, find center of mass

Moment about x axis  mass: $\rho f(x)\Delta x$ distance $\frac{1}{2} f(x)$

$$M_x = \rho \int_a^b \frac{1}{2} f(x)^2\,dx$$

Moment about y axis: mass $\rho f(x)\Delta x$ dist x

$$M_y = \rho \int_a^b x f(x)\,\Delta x.$$

Theorem The center of mass (aka centroid) is at $(\bar{x}, \bar{y})$ where:

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2} f(x)^2\,dx$$

Rmk Between 2 curves $y = f(x) \geq g(x)$ get

$$\bar{x} = \frac{1}{A}\int_a^b x\,(f(x) - g(x))\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2} f(x)^2 - \frac{1}{2} g(x)^2\,dx$$

Ex $y = x^2$ $x = y^2$ find centroid of region bounded

Probability

X

Continuous Random Variable: Things like height, battery life, blood pressure.
Each has a probability density function

Properties

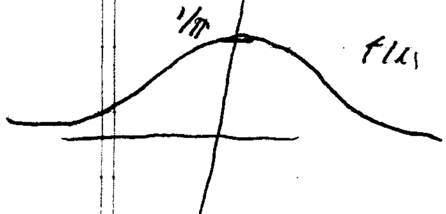
1. $f(x) \geq 0$ (no negative prob)
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (Total prob $= 1$)
3. $\int_a^b f(x)\,dx$ is prob that $a \leq X \leq b$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Example

Let $f(x) = \frac{c}{1+x^2}$. Choose c so this is density.
Find $P(-1 \leq x \leq 1)$

1. $\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = c\tan^{-1}\Big|_{-\infty}^{\infty} = \pi c$ so $c = 1/\pi$



2. $P(-1 \leq x \leq 1) = \int_{-1}^{1} \frac{1/\pi}{1+x^2}\,dx = \frac{1}{\pi}\tan^{-1}(x)\Big|_{-1}^{1}$

$$= \frac{1}{\pi}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = 1/2$$

Average = $\sum t \cdot \text{prob}(t)$

<u>Def</u> Let $f(x)$ be a prob density function. The <u>mean</u> is

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

<u>Rmk</u> Since $A=1$, this is just same as $\bar{x}$ calculation!

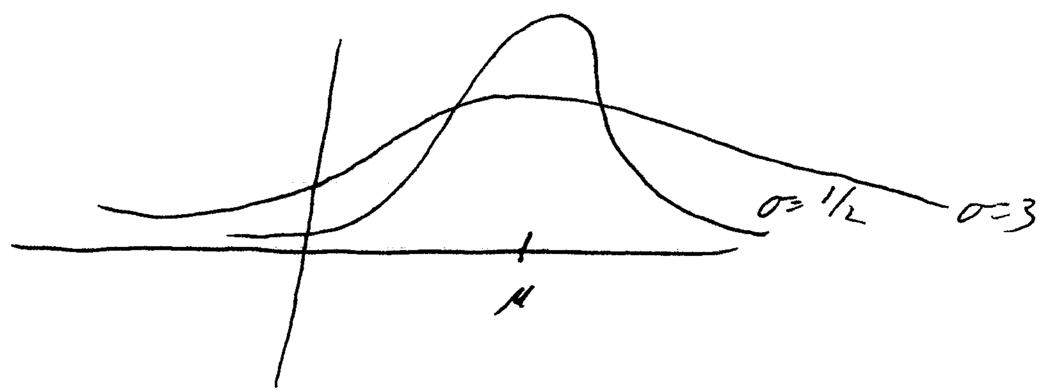
<u>Def</u> <u>Median</u> value means ½ higher & ½ lower.

To find median m set $\int_{m}^{\infty} f(x)\,dx = 1/2$ and solve for m.

<u>Example</u>

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$ is a density funct.

- mean = μ (obvious!)
- STD deviation = σ



"Bell curves"

Normal distributions

<u>Fact</u> Many Many X have normal distributions

Tables to look up integrals

Ex Exponential decreasing $f(t) = \begin{cases} 0 & t<0 \\ ce^{-ct} & t \geq 0 \end{cases}$

Models wait times, failure times, etc.

Ex p. 561 # 11