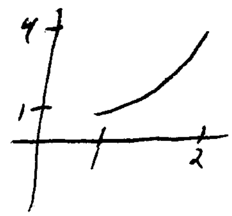
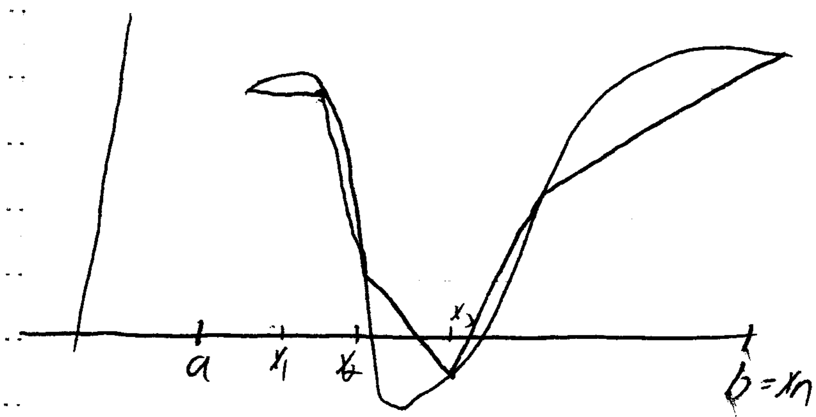
Lecture 12

Problem What is the length of the curve $y=x^2$ from $(1,1)$ to $(2,4)$?

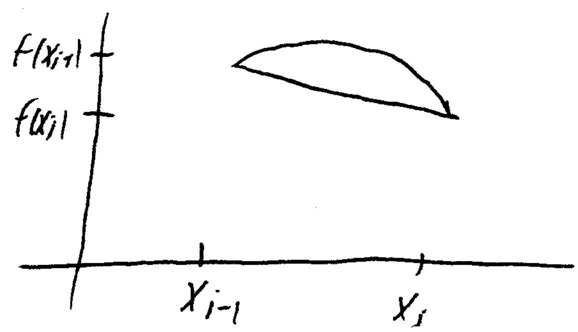


Procedure: Just like calculating areas or volumes, we come up with an approximation and let $n \to \infty$ to obtain an integral.
In this case we approximate by a sequence of line segments.

Suppose $y=f(x)$ for $a \le x \le b$:



Break into n line segments.



By Pythagorean Thm this line segment has length $\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$

Recall When x_{i-1} and x_i are very close, then

$$\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}} \approx f'(x_i) \text{ so}$$

$f(x_i)-f(x_{i-1}) \approx f'(x_i)(x_i-x_{i-1})$. Subbing in:

Length of line segment $\approx \sqrt{(x_i-x_{i-1})^2+f'(x_i)^2(x_i-x_{i-1})^2}$

$$= \sqrt{1+f'(x_i)^2}\,\Delta x$$ Thus:

Thm Suppose $f'(x)$ is continuous on $[a,b]$. Then the length of the curve $y=f(x)$ $a \le x \le b$ is:

$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Ex Problem above: $\int_1^2 \sqrt{1+(2x)^2}\,dx = \int_1^2 \sqrt{1+4x^2}\,dx$

$$= \sqrt{17} - \tfrac{1}{4}\ln(\sqrt{17}-4) - \tfrac{1}{2}\sqrt{5} - \tfrac{1}{4}\ln(2+\sqrt{5})$$
$$\approx 3.1678409$$

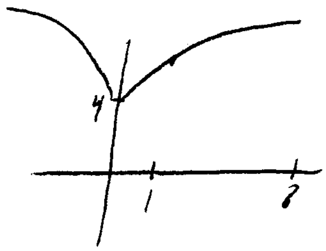
can do using $2x = \tan\theta$ substitution or tables.

Remark Integrals which arise in computing arclength are often difficult, so sometimes problems have strange functions!

Ex $y = 1+6x^{3/2}$ for $0 \le x \le 1$. $\frac{dy}{dx} = 9x^{1/2}$ so

$$A.L. = \int_0^1 \sqrt{1+81x}\,dx = \tfrac{2}{243}(1+81x)^{3/2}\Big|_0^1$$
$$= \tfrac{2}{243}\left(82^{3/2}-1\right)$$

Ex Find length of $x^2=(y-4)^3$ from $(1,5)$ to $(8,8)$



Notice $x=\pm(y-4)^{3/2}$ we are on curve $x=(y-4)^{3/2}$

So $g(y)=(y-4)^{3/2}$ $\quad g'(y)=\frac{3}{2}(y-4)^{1/2}$ $\quad g'(y)^2=\frac{9}{4}(y-4)$

$$A.L.=\int_5^8\sqrt{1+\tfrac{9}{4}(y-4)}\,dy=\int_5^8\sqrt{\tfrac{9}{4}y-8}\,dy=\tfrac{8}{27}\left(\tfrac{9}{4}y-8\right)^{3/2}\Big|_5^8$$

Ex. Use simpsons rule for $n=10$ to estimate length of $y=xe^{-x}$ $\quad 0\le x\le 5$

A $\quad y'=-xe^{-x}+e^{-x}$ so $AL=\int_0^5\sqrt{1+(-xe^{-x}+e^{-x})}\,dx$

$f(x)=1+(-xe^{-x}+e^{-x})$ $\quad x_0=0$ $x_1=\frac{1}{2}$ $x_2=1$ ... $x_9=4.5$ $x_{10}=5$

$$AL\approx\tfrac{.5}{3}\big(f(0)+4f(.5)+2f(1)+4f(1.5)+2f(2)+4f(2.5)+2f(3)+4f(3.5)+2f(4)+4f(4.5)+f(5)\big)$$

= ? (do on calculator!)

Ex. $y=\ln(\sec x)$ $\quad 0\le x\le \pi/4$ $\quad$ Find A.L.

A: $\frac{dy}{dx}=\frac{1}{\sec x}\cdot\sec x\tan x=\tan x$

$$A.L.=\int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx=\int_0^{\pi/4}\sec x\,dx=\ln|\sec x+\tan x|\Big|_0^{\pi/4}$$

$=\ln(\sqrt{2}+1)-\ln(1)$

$=\ln(1+\sqrt{2})$

Ex. Find a formula for the length of $y = x^2 - \frac{1}{8}\ln x$ from $(1,1)$ to point $(x, f(x))$.

Call this arc length function $S(x)$ so

$$S(x) = \int_1^x \sqrt{1 + f'(t)^2}\, dt \qquad f'(t) = 2t - \frac{1}{8t}$$

So

$$S(x) = \int_1^x \sqrt{1 + 4t^2 - \tfrac{1}{2} + \frac{1}{64t^2}}\, dt$$

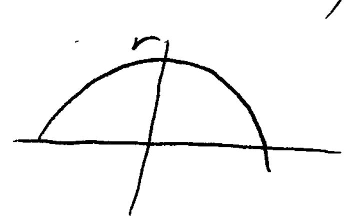
$$= \int_1^x \sqrt{4t^2 + \tfrac{1}{2} + \frac{1}{64t^2}}\, dt = \int_1^x \sqrt{\left(2t + \frac{1}{8t}\right)^2}\, dt$$

$$= \int_1^x 2t + \frac{1}{8t}\, dt = t^2 + \tfrac{1}{8}\ln t \Big|_1^x$$

$$= x^2 + \tfrac{1}{8}\ln x - 1$$

So $S(x) = x^2 + \frac{1}{8}\ln x - 1$ calculates arc length from $(1,1)$ to $\left(x, x^2 - \frac{1}{8}\ln x\right)$

Ex $y = \sqrt{r^2 - x^2}$ for $-r \le x \le r$



$$y' = \frac{1}{2\sqrt{r^2 - x^2}} \cdot -2x = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$AL = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

$$= \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx = r \cdot \sin^{-1}\left(\frac{x}{r}\right)\Big|_{-r}^{r}$$

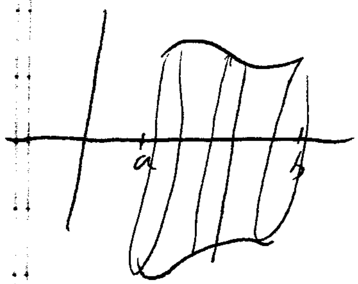
$$= r(\sin^{-1} 1 - \sin^{-1} -1)$$

$$= r(\pi/2 - -\pi/2) = \pi r$$

$\boxed{\text{Thus circumference is } 2\pi r}$

Areas of Revolution

Problem Suppose $y = f(x) \geq 0$ for $a \leq x \leq b$ is rotated about x axis what is area?



- Divide area into bands and unfold
- Circumference $= 2\pi r = 2\pi f(x)$
  Width = small piece of arclength
  $\approx \sqrt{1+f'(x_i)^2}\,\Delta x$

Thm Suppose $f(x) \geq 0$ and $f'(x)$ is continuous for all $a \leq x \leq b$. Then the surface area obtained by rotating the curve about the x axis is:

$$SA = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

Rmk Similarly for $x = g(y)$ $c \leq y \leq d$ about y axis.

Ex $y = \sqrt{4-x^2}$ $-1 \leq x \leq 1$ about x axis

$y' = \frac{-x}{\sqrt{4-x^2}}$ so $SA = 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx$

$= 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{\frac{4}{\sqrt{4-x^2}}}\,dx = 2\pi \int_{-1}^{1} 2\,dx = 8\pi$

** -1 to 1 same as $1-3$ etc....

Orange Peel Thm!