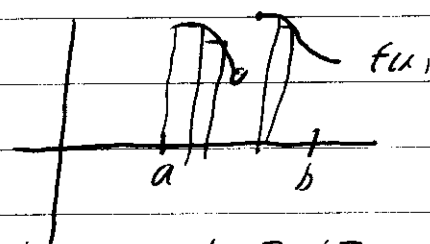
Lecture 9

Review Definite integral: $\int_a^b f(x)\,dx$ was limit of Riemann sums:

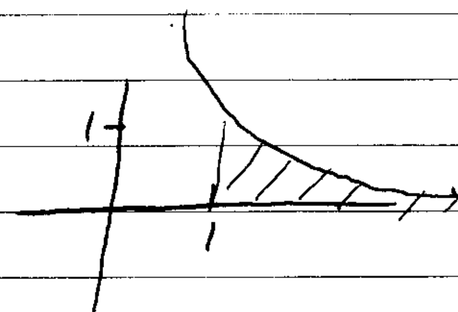


We assumed closed interval $[a,b]$ and assumed $f(x)$ has finitely many jump discontinuities, no vertical asymptotes

Today
- What about infinite intervals, ex $\int_1^\infty \frac{1}{x^2}\,dx$?
- What about vertical asymptotes ex $\int_1^3 \frac{1}{\sqrt{x-1}}\,dx$?

First Case

Consider $\int_1^\infty \frac{1}{x^2}\,dx$



should be shaded area.

Def $A(t) = \text{area above } [1,t] = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t} + 1$
$= 1 - \frac{1}{t}$

As $t \to \infty$, $A(t) \to 1$ so it seems total area is 1.

Write:
$$\int_1^\infty 1/x^2\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - 1/t$$
$$= 1$$

Formal Def

1. If $\int_a^t f(x)\,dx$ exists for all $t \geq a$ then define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \text{if it exists}$$

2. If $\int_t^b f(x)\,dx$ exists for all $t \leq b$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to -\infty} \int_t^b f(x)\,dx \quad \text{if it exists}$$

* Called <u>improper integrals</u>, if limit exists say they <u>converge</u>, if not they <u>diverge</u>

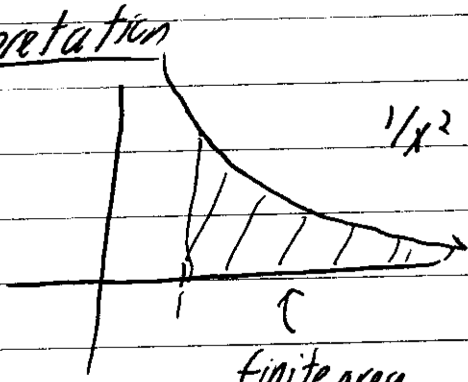
3. $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \quad \text{can choose any } a.$$

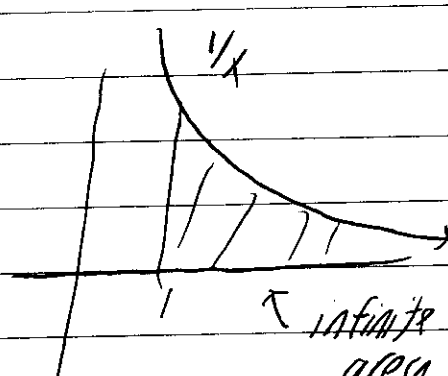
<u>Ex</u> 1. $\int_1^\infty 1/x^2\,dx$ converges (and $=1$)

2. $$\int_1^\infty 1/x\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left(\ln|x|\Big|_1^t\right) = \lim_{t\to\infty} \ln t = \infty$$

so $\int_1^\infty \frac{1}{x}\,dx$ diverges

<u>Interpretation</u>





3. $\int_1^\infty \frac{1}{x^{1.1}}dx = \lim_{t\to\infty} \int_1^t x^{-1.1}dx = \lim_{t\to\infty} \left(-10x^{-.1}\right)\Big|_1^t$

$= \lim_{t\to\infty} -10\left(\frac{1}{\sqrt[10]{t}} - 1\right) = 10$ converges.

<u>Exercise</u>

$\int_1^\infty \frac{1}{x^p}dx$ converges for $p>1$, diverges for $p\le 1$

4. $\int_{-\infty}^\infty \frac{x}{1+x^2}dx = \int_{-\infty}^0 \frac{x}{1+x^2}dx + \int_0^\infty \frac{x}{1+x^2}dx$

$= \frac{1}{2}\ln|1+x^2|\Big|_{-\infty}^0 + \frac{1}{2}\ln|1+x^2|\Big|_0^\infty$ diverges

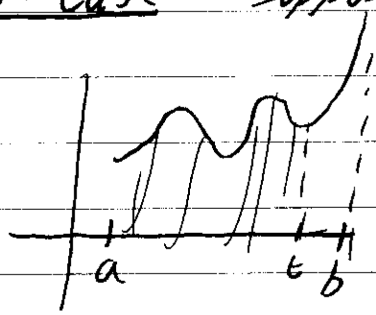
5 $\int_0^\infty se^{-3s}ds \underset{\text{parts}}{=} \left(-\frac{1}{3}se^{-3s} - \frac{1}{9}e^{-3s}\right)\Big|_0^\infty$ ← shorthand for $\lim_{t\to\infty}$

$= \left(\frac{-s}{3e^{+3s}} - \frac{1}{9e^{+3s}}\right)\Big|_0^\infty$

use that $\lim_{s\to\infty} \frac{s}{e^{3s}} = 0$ by LHR

$= 0 - \left(0 - \frac{1}{9}\right) = 1/9$

<u>Second Case</u> Suppose $f(x)$ continuous on $[a,b)$ w/ vert asy. at $x = b$



• Find $\int_a^t f(x)\,dx$

• Let $t \to b^-$, see if it converges

<u>Formal Def</u>

1. Suppose f cont on $[a,b)$ Then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx \quad \text{if it exists}$$

2. Suppose f cont on $(a,b]$. Then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

As before say converge or diverge.

<u>Ex</u>

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} 2\sqrt{x}\,\Big|_t^1$$

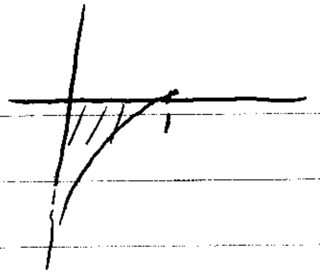
$$= \lim_{t \to 0^+} (2 - 2\sqrt{t}) = 2$$

converges!

<u>Exercise</u> $\int_0^1 \frac{1}{x^p}\,dx$ converges for $0 \leq p < 1$ and diverges $p \geq 1$.

opposite of case for $\int_1^\infty \frac{1}{x^p}\,dx$.

<u>Ex</u> $\int_0^1 \ln x\,dx$



Recall $\int \ln u\,du = u\ln u - u$ using parts

$$= \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \lim_{t\to 0^+} (x\ln x - x)\Big|_t^1$$

$$= \lim_{t\to 0^+} (-1) - (t\ln t - t)$$

$$= -1 - \lim_{t\to 0^+} t\ln t$$

Now $\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{LHR}{=} \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} -t = 0$

Thus $\int_0^1 \ln x\,dx = -1$, shaded area above is $\underline{1}$.

<u>Ex</u> $\int_{-2}^{3} \frac{1}{x^4}\,dx = \frac{-1}{3x^3}\Big|_{-2}^{3} = \left(\frac{-1}{81} - \frac{-1}{-24}\right) = -1/81 - 1/24$

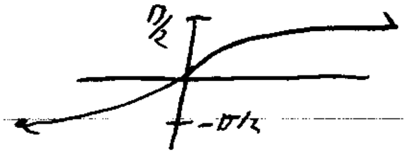
How can this be negative if $\frac{1}{x^4} > 0$?

WRONG! $\frac{1}{x^4}$ not continuous at $x=0$, FTOC does not apply.

$$\int_{-2}^{3} \frac{1}{x^4}\,dx = \int_{-2}^{0} \frac{1}{x^4}\,dx + \int_{0}^{3} \frac{1}{x^4}\,dx$$

both diverge

Review: Graph of $\tan^{-1}x$



$$\lim_{x\to\infty} \tan^{-1}x = \pi/2, \qquad \lim_{x\to-\infty} \tan^{-1}x = -\pi/2$$

Problem

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx \qquad u = e^x \quad du = e^x\,dx$$

$x = 0 \rightarrow u = 1$

$x = \infty \rightarrow u = \infty$

$$= \int_1^\infty \frac{du}{u^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^\infty$$

$$= \frac{1}{\sqrt{3}}\left(\pi/2 - \tan^{-1}(1/\sqrt{3})\right)$$

$$= 1/\sqrt{3}\,(\pi/2 - \pi/6)$$

$$= \frac{\pi}{3\sqrt{3}}$$