Name: SOLUTIONS

Math 142- Midterm Exam #3 - April 23, 2009

1. **(10 points)** Complete the following definitions precisely:

a. A sequence $\{a_n\}$ has the **limit** L, and we write $\lim_{n\to\infty} a_n = L$ if $\cdots$.

For any $\varepsilon > 0$ there is an N such that $|a_n - L| < \varepsilon$ whenever $n > N$.

b. Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

we say the series is **convergent** if $\cdots$.

the sequence of partial sums $\{s_n\}$ converges

where

$s_n = a_1 + a_2 + \cdots + a_n$

2. **(10 points)** Decide if the following series converge. If they do, then evaluate the sum.

a.

$$3 - 3/2 + 3/4 - 3/8 + 3/16 - 3/32 + \cdots.$$

Geometric $a = 3$ $r = -1/2$

$$\frac{3}{1 - -1/2} = \frac{3}{\frac{3}{2}} = 2$$

b.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}.$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$1 = A(n+2) + Bn$

$n = 0 \Rightarrow A = 1/2$

$n = -2 \Rightarrow B = -1/2$

$$\sum_{n=1}^{\infty} \frac{1/2}{n} - \frac{1/2}{n+2}$$

$$= 1/2\left(1 - \cancel{1/3} + 1/2 - \cancel{1/4} + \cancel{1/3} - \cancel{1/5} + \cancel{1/4} - 1/6 + \cancel{1/5} - 1/7 + \cdots\right)$$

telescoping

$$= \frac{1}{2}(1 + 1/2) = 3/4$$

3. **(30 points)** Decide whether the series converges absolutely, converges conditionally, or diverges. Justify your answer with the appropriate tests.

a.

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

converges absolutely by ratio test

b.

$$\sum_{n=1}^{\infty} \frac{2}{\pi^n + 3}$$

$$0 < \frac{2}{\pi^n+3} < \frac{2}{\pi^n}$$

and $\sum \frac{2}{\pi^n}$ converges (geometric, $r = 1/\pi$)

Thus this converges absolutely by comparison test c.

$$\sum_{n=2}^{\infty} \frac{n^5}{2^n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{2n^5} = 1/2$$

converges absolutely by ratio test

d.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+15}$$

converges by A.S.T.

But not absolutely by L.C.T. to $\Sigma 1/n$

Thus it converges conditionally

e.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}.$$

Diverges by L.C.T.
with $\sum \frac{1}{n}$

f.

$$1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots. \quad = \sum_{n=0}^{\infty} (-1)^n \frac{1.3.5 \ldots (2n-1)}{(2n-1)!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1.3.5 \ldots 2n-1 \cdot 2n+1}{(2n+1)!} \cdot \frac{(2n-1)!}{1.3.5 \ldots 2n-1}$$

$$= \lim_{n\to\infty} \frac{2n+1}{(2n+1)(2n)} = 0$$

converges abs
by ratio test

4. **(15 points)** True or false:

F a. Every bounded sequence converges.

T b. If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

T c. If $\sum c_n x^n$ diverges for $x = 6$ then it diverges for $x = 10$.

F d. The Ratio Test can be used to determine if $\sum 1/n^3$ converges.

T e. $1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$ diverges.

F f. If $0 \le a_n \le b_n$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

5. **(5 points)** Let $f(x) = 1/x^2$. Find the Taylor polynomial $T_3(x)$ centered at $a = 2$.

$f(2) = 1/4$

$f'(x) = \frac{-2}{x^3}$ $\quad f'(2) = -1/4$

$f''(x) = \frac{6}{x^4}$ $\quad f''(2) = 3/8$

$f'''(x) = \frac{-24}{x^5}$ $\quad f'''(2) = -24/32 = -3/4$

$$1/4 + -1/4(x-2) + \frac{3/8}{2!}(x-2)^2 - \frac{3/4}{3!}(x-2)^3$$

$$= 1/4 - 1/4(x-2) + 3/16(x-2)^2 - 1/8(x-2)^3$$

6. **(10 points)** Find the interval of convergence:

$$\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n(x-3)^n}\right|$$

$$= 2(x-3)\frac{\sqrt{n+3}}{\sqrt{n+4}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-3|$$

so $-1 < 2x-6 < 1$

$5 < 2x < 7$

$5/2 < x < 7/2$

$x = 5/2$ $\quad \sum_{n=0}^{\infty} \frac{2^n(-1/2)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

conv by AST

$x = 7/2$ $\quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ div

$[5/2, 7/2)$

7. **(10 points)** Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (from the definition) or known series.

a. $f(x) = \frac{x^2}{1+x}$.

b. $f(x) = xe^{2x}$.

a. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \ldots$

$\frac{x^2}{1+x} = x^2 - x^3 + x^4 - x^5 + x^6 - \ldots \quad R = 1$

b. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty$

$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1} \quad R = \infty$

8. **(10 points)** Use series to approximate:

$$\int_0^1 x\cos(x^3)dx$$

with an error at most 1/1000.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -$$

$$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$$

$$x\cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \cdots$$

$$\int_0^1 x\cos(x^3) = \frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{16} + \frac{1}{336} - \frac{1}{14400}$$

$$\begin{array}{r} 24 \\ 14 \\ 96 \\ 240 \\ \hline 336 \end{array}$$

Thus

$$\frac{1}{2} - \frac{1}{16} + \frac{1}{336}$$

has $|\text{error}| < \frac{1}{14400}$