Name: SOLUTIONS

# Math 142- Midterm Exam #2 - March 24, 2009

1. **(10 points)** a. Use the trapezoidal rule with $n = 4$ to approximate the integral $\int_0^2 x^3 dx$.

b. Recall the error bound for the trapezoidal rule, if $|f''(x)| \leq K$ for $a \leq x \leq b$ then the error:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Use this to obtain an error bound for your approximation.

a. $\Delta x = 1/2$, $n = 4$

$$T_4 = \frac{\Delta x}{2}\left(f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + f(2)\right)$$

$$= 1/4\left(0 + 2 \cdot 1/8 + 2 \cdot 1 + 2 \cdot 27/8 + 8\right)$$

$$= 1/4\left(1/4 + 2 + 27/4 + 8\right) = 17/4$$

b $f''(x) = 6x$ so for $0 \leq x \leq 2$ we have $|f''(x)| \leq 12$, $K = 12$

$$E_T \leq \frac{12(2^3)}{12 \cdot 4^2} = 1/2$$

2. **(10 points)** Determine if the integral converges or diverges, and if it converges, evaluate it.

a. $\int_0^\infty \frac{2x}{(x^2+3)^2}\,dx$

$$u = x^2+3 \quad du = 2x\,dx$$

$$= \int_3^\infty \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_3^\infty = (0 - -1/3) = 1/3$$

converges

b. $\int_0^{28}(x-1)^{-1/3}dx$

$$= \int_0^1 (x-1)^{-1/3}dx + \int_1^{28} (x-1)^{-1/3}dx$$

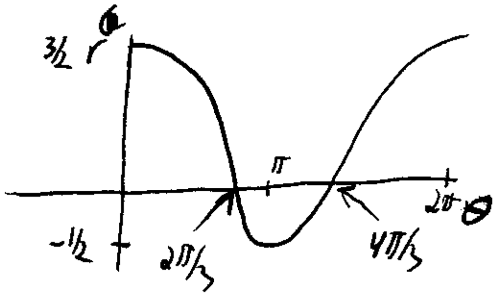
$$= \frac{3}{2}(x-1)^{2/3}\Big|_0^1 + \frac{3}{2}(x-1)^{2/3}\Big|_1^{28}$$

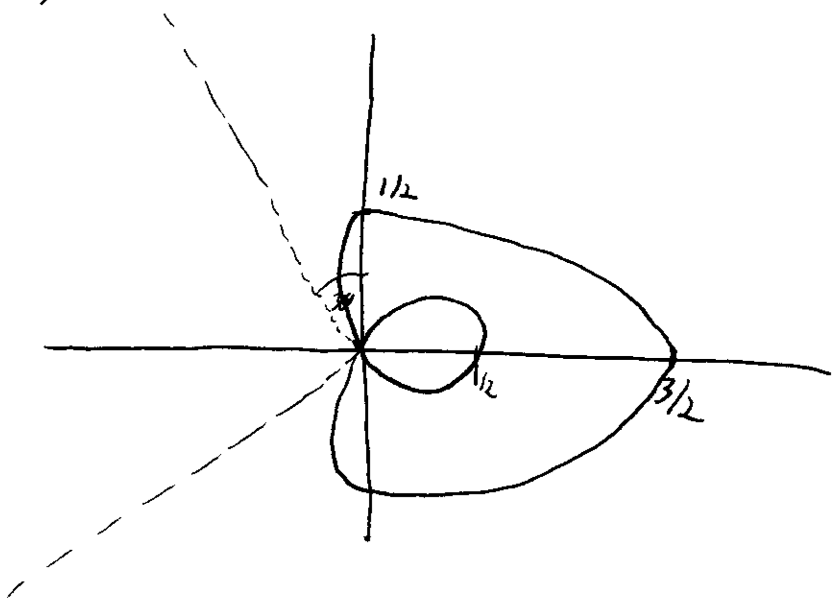
$$= \frac{3}{2}(0-1) + \frac{3}{2}(9-0)$$

$$= -3/2 + 27/2 = 12$$

3. **(10 points)**

a. Sketch the polar curve $r = \frac{1}{2} + \cos\theta$. Hint: $\cos(2\pi/3) = \cos(4\pi/3) = -1/2$.




b. Set up but do not evaluate an integral that gives the area of the inner loop.

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2 \, d\theta$$

$$= \boxed{\int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta}$$

4. **(10 points)** Let

$$x = 1 + e^t, y = t^3 + 2, 1 \leq t \leq 3$$

be a parametric curve. Set up, but do not evaluate, an integral that represents the length of the curve.

$$\int_1^3 \sqrt{(e^t)^2 + (3t^2)^2}\, dt$$

5. **(10 points)** Let $x = 6e^t, y = t^2$ be a parametric curve. Find the equation of the tangent line to this curve at the point $(6e^2, 4)$. Is the curve concave up or concave down at this point?

$$\frac{dy}{dx} = \frac{2t}{6e^t} \qquad (6e^2, 4) \text{ is when } t = 2$$

$$\text{slope} = \frac{4}{6e^2} \qquad \text{point} = (6e^2, 4)$$

$$\boxed{y - 4 = \frac{4}{6e^2}(x - 6e^2)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{12e^t - 12te^t}{36e^{2t}}}{6e^t}$$

$$= \frac{12e^t(1 - 12t)}{216e^{3t}}$$

when $t = 2$ this is $< 0$ so

concave ~~up~~ down

6. **(10 points)** The length of time spent waiting in a bank is modeled with an exponential density function with mean 8 minutes. (Recall such a density function has the form $f(t) = ce^{-ct}$, $0 \le t < \infty$.)

a. What is the probability a customer must wait more than 10 minutes?
b. What is the median waiting time?

$c = 1/\text{mean} = 1/8$ so $f(t) = 1/8\, e^{-t/8}$

a. $\int_{10}^{\infty} 1/8\, e^{-t/8}\, dt = -e^{-t/8} \Big|_{10}^{\infty}$

$= (0 - -e^{-10/8}) = e^{-5/4}$

b. $\int_{m}^{\infty} \frac{1}{8} e^{-t/8}\, dt = 1/2$, solve for $m$

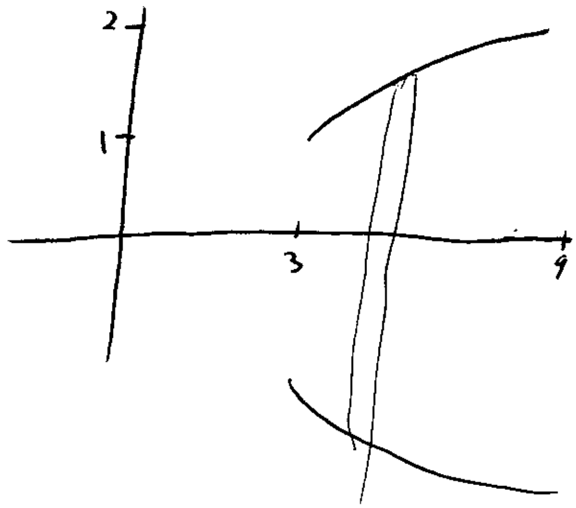
$\parallel$

$0 - -e^{-m/8}$ so $e^{-m/8} = 1/2$

$-m/8 = \ln(1/2)$

$m = -8\ln(1/2)$

7. **(10 points)** Consider the curve $x = 1 + 2y^2$ for $1 \le y \le 2$. Find the area of the surface that results when this curve is rotated about the $x$ axis.




$$A = \int 2\pi r\, ds \qquad r = y \qquad ds = \sqrt{1+g'(y)^2}\, dy$$

$$= \sqrt{1+16y^2}\, dy$$

$$A = \int_1^2 2\pi y \sqrt{1+16y^2}\, dy$$

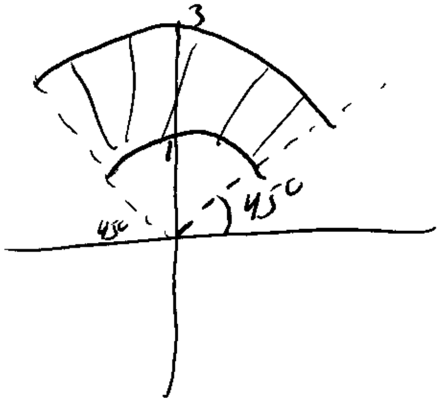
$$u = 1+16y^2 \qquad du = 32y\, dy$$

$$= \int \frac{\pi}{16} u^{1/2} dy = \frac{\pi}{24}\left(1+16y^2\right)^{3/2} \Big|_1^2$$

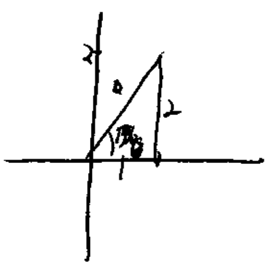
$$= \frac{\pi}{24}\left(\cancel{1} \, 65^{3/2} - 17^{3/2}\right)$$

8. **(10 points)**

a. Sketch the region given by the polar inequalities $1 \leq r \leq 3$ and $\pi/4 < \Theta < 3\pi/4$.




b. Consider the point $(1, 2)$ in the $xy$-plane. Give three different representations of this point in polar coordinates. At least one of your three should have $r < 0$.




$\theta = \tan^{-1}(2) \quad r = \sqrt{5}$

$(\sqrt{5}, \tan^{-1}(2))$

$(\sqrt{5}, \tan^{-1}(2) + 2\pi)$

$(-\sqrt{5}, \tan^{-1}(2) + \pi)$

9. (**10 points**) Consider the parametric equations below.

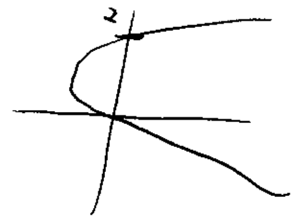
$$x = t^2 - 1, y = 1 - t, \ -2 \le t \le 3.$$

a. Eliminate the parameter to find a Cartesian equation of the curve.
b. Sketch the curve. Indicate with an arrow the direction in which the curve is traced as t increases.

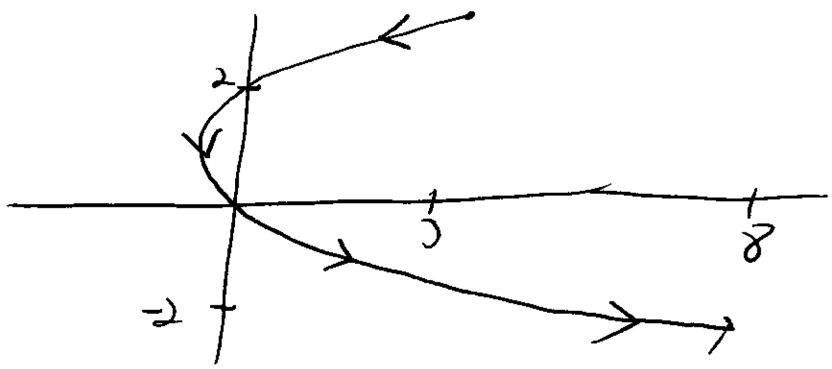
a. $t = 1 - y$ $\quad x = (1-y)^2 - 1$

$= y^2 - 2y + 1 - 1$

$\boxed{x = y^2 - 2y}$




b Go from $(3,3)$ to $(8,-2)$



**Write name again please:**

10. (**10 points**) Does the integral

$$\int_1^{\infty} \frac{1}{6x^7 + 33x^5 + 3x^2 + 82} dx$$

converge or diverge? Justify your answer.

$f(x)$ is clearly $\geq 0$ and $\leq \frac{1}{6x^7}$

But $\int_1^{\infty} \frac{1}{6x^7} = -\frac{1}{36} x^{-6} \Big|_1^{\infty}$

$= (0 - -1/36) = 1/36$

converges

This this integral converges, by the comparison thm.