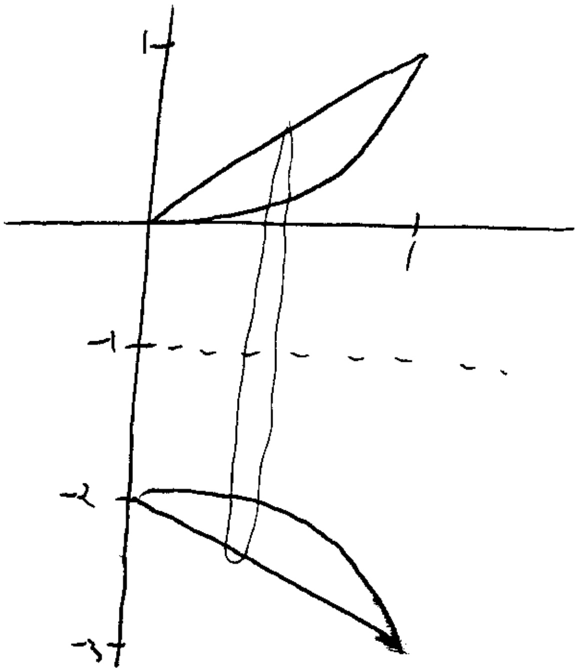
Name: SOLUTIONS

Math 142- Midterm Exam #1 - February 17, 2009

Directions: The exam is worth 100 points. You may not use any calculators, notes or study aides of any kind.

1. **(10 points)** Consider the region enclosed by the curves $y = x^3$ and $y = x$ for $x \geq 0$. Find the volume of the solid obtained by rotating this region around the line $y = -1$.



using washers:

small radius $= 1 + x^3$

large radius $= 1 + x$

$A(x) = \pi(1+x)^2 - \pi(1+x^3)^2$

$$V = \int_0^1 \pi(1+x)^2 - \pi(1+x^3)^2 \, dx$$

$$= \pi \int_0^1 x^2 + 2x + 1 - x^6 - 2x^3 - 1 \, dx$$

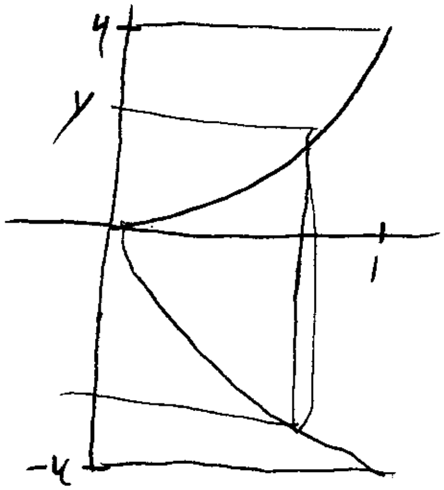
$$= \pi \int_0^1 x^2 + 2x - x^6 - 2x^3 \, dx$$

$$= \pi \left[\tfrac{1}{3}x^3 + x^2 - \tfrac{1}{7}x^7 - \tfrac{1}{2}x^4 \right]_0^1$$

$$= \pi \left(1/3 + 1 - 1/7 - 1/2 \right)$$

$$= \pi \left(\frac{14 + 42 - 6 - 21}{42} \right) = 29\pi/42$$

2. (**10 points**) Consider the curves $y = 4x^2$, $x = 0$, $y = 4$. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.



Integrate for $0 \le y \le 4$

Typical shell as radius y

height $= x = \frac{1}{2}\sqrt{y}$

$$V = \int_0^4 2\pi y \cdot \tfrac{1}{2}\sqrt{y}\, dy$$

$$= \pi \int_0^4 y^{3/2}\, dy$$

$$= \pi \cdot \frac{2}{5} y^{5/2} \Big|_0^4$$

$$= 64\pi/5$$

3. **(10 points)** Let

$$f(x) = xe^{\frac{x}{2}}.$$

Find the average value of $f(x)$ on $[1, 3]$.

$\int x e^{x/2}\,dx$

$u = x \quad v = 2e^{x/2}$

$du = dx \quad dv = e^{x/2}\,dx$

$= 2xe^{x/2} - \int 2e^{x/2}\,dx = 2xe^{x/2} - 4e^{x/2}$

$$f_{ave} = \frac{1}{3-1}\int_1^3 f(x)\,dx$$

$$= \frac{1}{2}\left(2xe^{x/2} - 4e^{x/2}\right)\Big|_1^3$$

$$= \left(xe^{x/2} - 2e^{x/2}\right)\Big|_1^3$$

$$= \left(3e^{3/2} - 2e^{3/2}\right) - \left(e^{1/2} - 2e^{1/2}\right)$$

$$= e^{3/2} + e^{1/2}$$

4. **(10 points)** Find the area beneath the graph of $y = x^2 \sin(2x)$ and above the x axis for $0 \le x \le \pi/2$.

$$\int_0^{\pi/2} x^2 \sin(2x)\,dx$$

$$u = x^2 \qquad v = -\tfrac{1}{2}\cos(2x)$$
$$du = 2x\,dx \qquad dv = \sin(2x)\,dx$$

$$= -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\,dx$$

$$u = x \qquad v = \tfrac{1}{2}\sin(2x)$$
$$du = dx \qquad dv = \cos(2x)\,dx$$

$$= \frac{-x^2}{2}\cos(2x) + \tfrac{1}{2}x\sin(2x) - \int \tfrac{1}{2}\sin(2x)\,dx$$

$$= \frac{-x^2}{2}\cos(2x) + \tfrac{1}{2}x\sin(2x) + \tfrac{1}{4}\cos(2x)\Big|_0^{\pi/2}$$

use $\cos 0 = 1$ $\cos(\pi) = -1$
$\sin 0 = 0 = \sin \pi$

$$= \left(\frac{\pi^2}{8} + 0 - 1/4\right) - (0 + 0 + 1/4) = \frac{\pi^2}{8} - \frac{1}{2}$$

$$= \cancel{\frac{\pi^2}{8} - \frac{\pi}{4} - \frac{1}{4}}$$

5. (**15 points**) Evaluate the following integral. Hint: You will eventually need $\int \sec t\, dt = \ln|\sec t + \tan t|$.

$$\int \frac{x}{\sqrt{x^2+2x+5}}dx$$

$x^2+2x+5 = (x+1)^2+4$ $\quad u = x+1 \quad du = dx$

$x = u-1$

$$= \int \frac{u-1}{\sqrt{u^2+4}}du = \int \frac{u}{(u^2+4)^{1/2}}du - \int \frac{1}{\sqrt{u^2+4}}du$$

$$= (u^2+4)^{1/2} - \int \frac{1}{\sqrt{u^2+4}}du$$

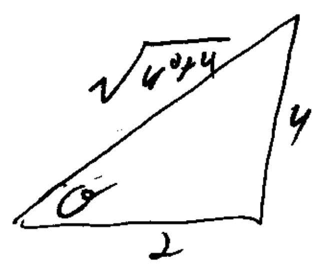
set $u = 2\tan\theta \quad du = 2\sec^2\theta\, d\theta$

$$-\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta+4}}$$

$$-\int \frac{2\sec^2\theta\, d\theta}{2\sec\theta}$$

$$-\int \sec\theta\, d\theta$$

$$= (u^2+4)^{1/2} - \ln|\sec\theta+\tan\theta|$$



$\sec\theta = \frac{\sqrt{u^2+4}}{2}$

$$= (u^2+4)^{1/2} - \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C$$

$$\boxed{= \sqrt{x^2+2x+5} - \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C}$$

6. **(15 points)**

$$\int \frac{3x^2 + 12x + 29}{(x^2 + 4x + 13)(x + 3)}dx.$$

$$\frac{3x^2+12x+29}{(x^2+4x+13)(x+3)} = \frac{Ax+B}{x^2+4x+13} + \frac{C}{x+3}$$

$$3x^2+12x+29 = (Ax+B)(x+3) + C(x^2+4x+13)$$

$$x=-3 \longrightarrow 20 = 10C \qquad C = 2$$

$$x^2 \text{ coef}: \rightarrow 3 = A + C \longrightarrow A = 1$$

$$\cancel{\text{co}}\; x=0 \longrightarrow 29 = 3B + 13C \longrightarrow B = 1$$

$$\int \frac{x+1}{x^2+4x+13} + \frac{2}{x+3}\,dx = \int \frac{x+1}{(x+2)^2+9}dx + 2\ln|x+3|$$

$$u = x+2 \quad du = dx$$

$$= \int \frac{u-1}{u^2+9}du + 2\ln|x+3|$$

$$= \int \frac{u}{u^2+9}du - \frac{1}{u^2+9}du + 2\ln|x+3|$$

$$= \frac{1}{2}\ln|u^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + 2\ln|x+3|$$

$$\boxed{= \frac{1}{2}\ln|x^2+4x+13) - \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + 2\ln|x+3| + C}$$

7. **(10 points)**

$$\int \frac{x+1}{(x-2)^2} dx.$$

$$\frac{x+1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$x+1 = A(x-2) + B$

$x=2 \rightarrow 3 = B$

$x=0 \rightarrow 1 = -2A + B \rightarrow A = 1$

$$\int \frac{1}{x-2} + \frac{3}{(x-2)^2}\, dx =$$

$$\boxed{\ln|x-2| - \frac{3}{x-2} + C}$$

8. **(10 points)**

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx.$$

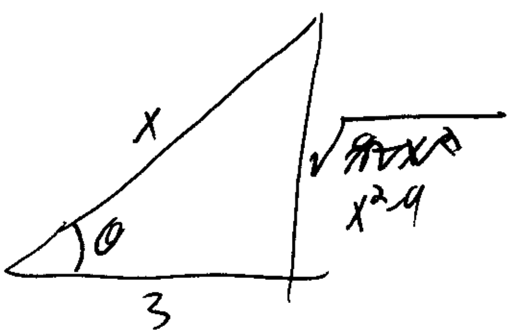
$x = 3\sec\theta \quad dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{3\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{9\sec^2\theta-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\cdot 3\tan\theta\,d\theta}$$

$$= \int \frac{1}{9\sec\theta}\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

$\sec\theta = x/3$



so $\sin\theta = \frac{\sqrt{9-x^2}}{x}$

Answer: $\boxed{\frac{\sqrt{x^2-9}}{9x} + C}$

Please write name again:

9. (**10 points**) $\int \sec^4 x \tan^2 x \, dx$.

$\int \sec^2 x \tan^2 x \sec^2 x \, dx$

$\int (\tan^2 x + 1) \tan^2 x \sec^2 x \, dx$

$u = \tan x \quad du = \sec^2 x \, dx$

$= \int (u^2+1) u^2 \, du$

$= u^5/5 + u^3/3 + C$

$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$