Name: SOLUTIONS

Math 142- Final Exam - May 7, 2009

Instructions: The exam is worth 150 points. Calculators are not permitted. You are permitted two 3×5 notecards.

1. **(10 points)** Compute the following integrals.

a. $\int \frac{1}{x^2+9}\, dx$.

$$\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

b. $\int \frac{1}{x^2-9}\, dx$.

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$1 = A(x+3) + B(x-3)$$

$x=3 \to A = 1/6$

$x=-3 \to B = -1/6$

$$\int \frac{1/6}{x-3} - \frac{1/6}{x+3}\, dx$$

$$= \frac{1}{6}\ln|x-3| - 1/6 \ln|x+3| + C$$

2. **(10 points)** Compute the following integral.

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx.$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

x^2 coef: $2 = A + B$

x coef: $-1 = C$

const: $4 = 4A$

$A = 1 \quad B = 1 \quad C = -1$

$$\int \frac{1}{x} + \frac{x}{x^2+4} \quad \frac{-1}{x^2+4}\, dx$$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

3. **(10 points)** Compute the following integrals.

a. $\int t\cos(t^2+5)\,dt$.

$u = t^2+5 \quad du = 2t\,dt$

$\int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + C$

$= \boxed{\frac{1}{2}\sin(t^2+5)+C}$

b. $\int t\cos(t+5)\,dt$.

$u = t \quad dv = \cos(t+5)\,dt$

$du = dt \quad v = \sin(t+5)$

$\int u\,dv = uv - \int v\,du$

$= t\sin(t+5) - \int \sin(t+5)\,dt$

$= t\sin(t+5) + \cos(t+5) + C$

4. **(10 points)** Compute the following integrals.

a. $\int \sec^3 x \tan^3 x \, dx$. use $\tan^2 x + 1 = \sec^2 x$

$= \int \sec^2 x \tan^2 x \sec x \tan x \, dx$

$= \int \sec^2 x(\sec^2 x - 1) \sec x \tan x \, dx$ $u = \sec x$ $du = \sec x \tan x \, dx$

$= \int u^2(u^2 - 1) du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

b. $\int \frac{\sqrt{9-x^2}}{x^2} dx$.

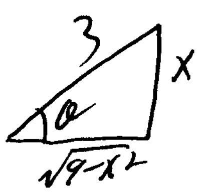
$x = 3\sin\theta$ $dx = 3\cos\theta \, d\theta$

$\int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta \, d\theta = \int \frac{3\cos\theta \, 3\cos\theta}{9\sin^2} d\theta$

$= \int \cot^2\theta \, d\theta$

$= \int \csc^2\theta - 1 \, d\theta$

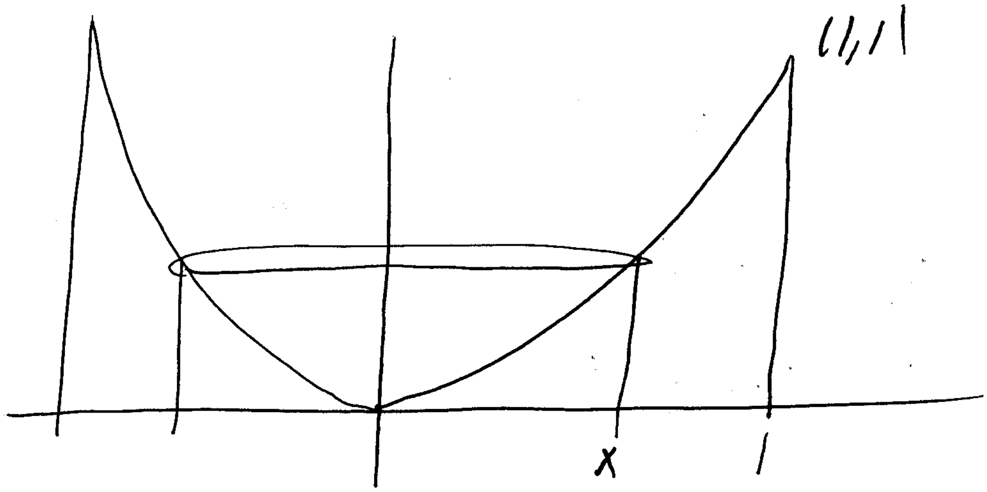
$= -\cot\theta - \theta + C$ Now



$\cot\theta = \frac{\sqrt{9-x^2}}{x}$ $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

Answer: $\frac{-\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

5. **(10 points)** Consider the region bounded by the curves $y = x^2$, $y = 0$ and $x = 1$. Use the method of cylindrical shells to find the volume obtained by rotating this region about the y-axis. Sketch the region and a typical shell.



radius $= x$

height $= x^2$

$$V = \int 2\pi r h \, dr = \int_0^1 2\pi x \cdot x^2 \, dx$$

$$= \int_0^1 2\pi x^3 \, dx = \frac{\pi x^4}{2} \Big|_0^1$$

$$= \pi/2$$

6. **(10 points)** Evaluate the following integrals.

a. $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$.

$$= \int_0^1 (x-1)^{-1/3}\,dx + \int_1^9 (x-1)^{-1/3}\,dx$$

$$= \frac{3}{2}(x-1)^{2/3}\Big|_0^1 + \frac{3}{2}(x-1)^{2/3}\Big|_1^9$$

$$= \frac{3}{2}(0-1) + \frac{3}{2}(4-0)$$

$$= -\frac{3}{2} + \frac{12}{2} = 9/2$$

must split into two pieces for credit.

b. $\int_{-\infty}^0 xe^x\,dx$.

$u = x \quad v = e^x$

$du = dx \quad dv = e^x dx$

$$= xe^x - \int e^x$$

$$= \lim_{m \to -\infty} \left(xe^x - e^x\right)\Big|_m^0$$

$$= \lim_{m \to -\infty} \left((-1) - (me^m - e^m)\right)$$

$$= -1$$

7. **(10 points)** Let $x = \frac{t}{1+t}$, $y = \ln(1+t)$ for $0 \le t \le 2$. ~~Find the exact length of this curve.~~ Set up integral

$$x' = \frac{1+t-t}{(1+t)^2} \qquad y' = \frac{1}{1+t}$$

$$AL = \int_0^2 \sqrt{\frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}}\, dt$$

8. **(10 points)** Sketch neatly the polar curve $r = -3\cos(\theta)$.

$$r^2 = -3r\cos\theta$$

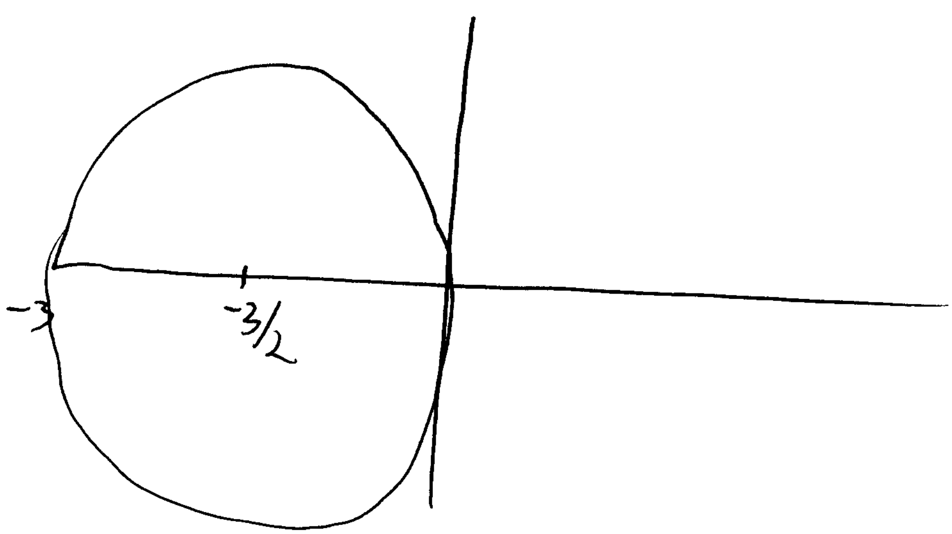
$$x^2 + y^2 = -3x$$

$$x^2 + 3x + y^2 = 0$$

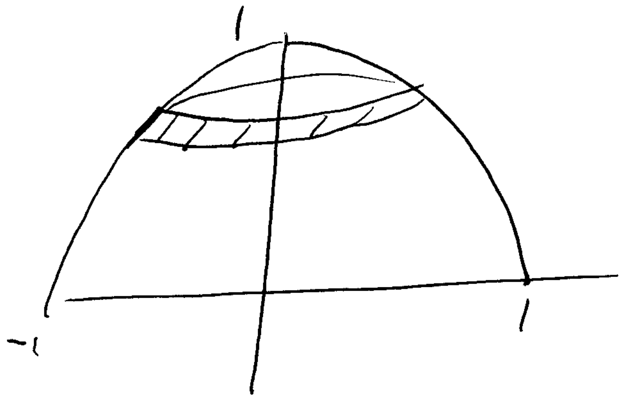
$$x^2 + 3x + 9/4 + y^2 = 9/4$$

$$(x + 3/2)^2 + y^2 = (3/2)^2$$

circle, center $(-3/2, 0)$ radius $3/2$



9. **(10 points)** Consider the curve $y = 1 - x^2$ for $0 \le x \le 1$. Find the *surface area* obtained by rotating this curve about the y-axis.



integrate $S.A. = \int 2\pi x \, ds$

$$ds = \sqrt{1 + (-2x)^2} = \sqrt{1+4x^2}$$

$$S.A. = \int_0^1 2\pi x \sqrt{1+4x^2} \, dx$$

$$u = 1 + 4x^2 \quad du = 8x \, dx$$

$$= \int_1^5 \frac{\pi}{4} \sqrt{u} \, du = \frac{\pi}{6} u^{3/2} \Big|_1^5$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

10. **(10 points)** Consider the parameterized curve $x = e^t + 2$, $y = t^2 + t + 1$. Find the equation of the tangent line to this curve at the point $(3, 1)$.

$x' = e^t \quad y' = 2t + 1$

$(3,1)$ occurs when $t = 0$

$$\frac{dy}{dx} = \frac{2t+1}{e^t} \qquad \text{slope} = \left.\frac{dy}{dx}\right|_{t=0} = \frac{1}{1} = 1$$

$$y - 1 = 1(x - 3)$$

$$y = x - 2$$

11. **(10 points)** Complete the following definitions precisely:

a. A sequence $\{a_n\}$ has the **limit** L, and we write $\lim_{n\to\infty} a_n = L$ if $\cdots$.

For any $\varepsilon > 0$ there is $N > 0$ such that

$|a_n - L| < \varepsilon$ whenever $n > N$.

b. Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

we say the series is **convergent** if $\cdots$.

The sequence of partial

sums converges

12. **(10 points)** Compute sin(1) and $1/e$ such that the absolute value of the error in each case is $< 1/1000$. Hint: Use series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \ldots$$

$< 1/1000$

$$1 - 1/6 + 1/120$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\frac{1}{e} = e^{-1} = \underbrace{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}}_{} - \frac{1}{5040}$$

$$e^{-1} \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$

13. **(10 points)** Find the Maclaurin series for

$$f(x) = \frac{x}{x^2+16}. \quad = \frac{1}{16} \; \frac{x}{\frac{x^2}{16}+1}$$

$$= \frac{x}{16} \cdot \frac{1}{1+\left(\frac{x}{4}\right)^2}$$

Now

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{1}{1+\left(\frac{x}{4}\right)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{2n}}$$

$$\frac{x}{16} \cdot \frac{1}{1+\left(\frac{x}{4}\right)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{4^{2n+2}}$$

$$= \cancel{16\,x - \frac{x^3}{1} + x^5}$$

$$\frac{x}{4^2} - \frac{x^3}{4^4} + \frac{x^5}{4^6} - \frac{x^7}{4^8} - \cdots$$

14. **(10 points)** a. Consider

$$\sum_{n-1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}.$$

Determine if this series diverges, converges conditionally or converges absolutely. Justify your answer.

Converges by A.S.T.

However $\sum \frac{n}{\sqrt{n^{3/2}}}$ diverges by L.C.T w/ $\sum \frac{1}{\sqrt{n}}$

Thus converges conditionally

b. Do the same for the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$.

$$0 \leq \left|\frac{\cos n}{n^2+1}\right| \leq \frac{1}{n^2+1}$$

Thus $\sum \frac{\cos n}{n^2+1}$ converges absolutely

by comparison test.

15. **(10 points)**

a. What is a geometric series? Under what circumstances is it convergent? What is its sum?

Geometric Series is of form $a + ar + ar^2 + \ldots = \sum_{n=0}^{\infty} ar^n$

Converges if & only if $-1 < r < 1$

If so, then sum is $\frac{a}{1-r}$.

b. Suppose $\sum_{n=0}^{\infty} a_n = 3$ and s_n is the n-th partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$

$\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} s_n = 3$