Lecture 19

Review We used implicit diff to calculate $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \ldots$

and $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Cor Let $y = x^n$. Then $y' = nx^{n-1}$. (n any real #, before we did $n = 1, 2, 3, \ldots$)

Proof $\ln y = n \ln x$

$$\frac{1}{y} y' = \frac{n}{x} \Rightarrow y' = \frac{ny}{x} = nx^{n-1} \quad //$$

Logarithmic Diff

Given $y = f(x)$

1. Take ln of both sides: $\ln y = \ln f(x)$
2. Diff implicitly $\frac{1}{y} \cdot y' = \text{—}$
3. Mult by y on both sides

When? For functions when taking ln simplifies

EX $y = \dfrac{e^{-x}\cos^2 x}{x^2+x+1}$

$$\boxed{y' = \frac{e^{-x}\cos^2 x}{x^2+x+1}\left(\quad \right)}$$

$$\ln y = -x + 2\ln\cos x - \ln(x^2+x+1)$$

$$\frac{1}{y} y' = -1 - \frac{2\sin x}{\cos x} - \frac{2x+1}{x^2+x+1}$$

Warning $\frac{d}{dx} b^x = b^x \ln b$ exponential, base is a constant

$\frac{d}{dx} x^n = n x^{n-1}$ power, exponent is a constant.

What about $f(x)^{g(x)}$? Logarithmic differentiation!

Ex

$$y = \cos x^x$$

$$\ln y = x \ln|\cos x|$$

$$\frac{1}{y} y' = \ln(\cos x) + x \cdot \frac{-\sin x}{\cos x}$$

$$y' = (\cos x)^x \left[\ln \cos x - \frac{x \sin x}{\cos x}\right]$$

Ex

$$y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = \ln x + 1 \qquad \boxed{y' = x^x(\ln x + 1)}$$

Ex $x^y = y^x$. Find y'

$$y \ln x = x \ln y$$

$$y' \ln x + \frac{y}{x} = \ln y + x \cdot \frac{y'}{y}$$

$$y'\left(\ln x - \frac{x}{y}\right) = \ln y - y/x$$

$$\boxed{y' = \frac{\ln y - y/x}{\ln x - x/y}}$$

Fun limit $f(x) = \ln x \qquad f'(x) = 1/x$

$$f'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\ln(1+x)$$

$$1 = \lim_{x\to 0} \ln\left((1+x)^{1/x}\right).$$ Now e^x is continuous so

$$e = \lim_{x\to 0}(1+x)^{1/x}$$ Let $n = \frac{1}{x}$

Conclude: $$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

$$2^2, \left(\frac{3}{2}\right)^3, \left(\frac{4}{3}\right)^4, \left(\frac{5}{4}\right)^5, \text{etc.} \to e$$

Exponential Growth

Recall Population growth $P(t)$ we expected $P'(t)$ to be proportional to $P(t)$!

Differential equation $\frac{dy}{dt} = ky$

Rmk $k > 0$ this is exponential growth
$k < 0$ " " " decay

Thm The only solution to $\frac{dy}{dt} = ky$ are exponential functions

$$y(t) = y(0)\,e^{kt}$$

constant is easily seen to be $y(0)$!

* k is growth/decay rate

<u>Observation</u> Assuming exponential growth/decay, 2 data points lets you find $y(0)$ and K.

Ex Bacteria culture grows w/ constant relative growth

400 after 2 hours

25000 after 6

a. Find k, $P(0)$

b. Find $P(t)$

c. # cells after 4 hours?

d. When are 50,000 bacteria present?

<u>Solution</u> $P(t) = P(0)e^{kt}$

$$400 = P(0)\, e^{2k}$$

$$25000 = P(0)\, e^{6k} \Rightarrow 62.5 = e^{4k}$$

$$\ln(62.5) = 4K$$

$$K = \frac{\ln(62.5)}{4} \approx 1.0338$$

$$400 = P(0)e^{2.0675}$$

$$P(0) = 50.6$$

$$\boxed{P(t) = 50.6\, e^{1.0338\, t}}$$

c. Plug in $t = 4$

d.

Radioactive Decay

$$m(t) = m(0)e^{kt} \qquad k < 0$$

Rmk When is $m(t) = \frac{1}{2}m(0)$?

$$\frac{1}{2}m(0) = m(0)e^{kt}$$

$$\frac{1}{2} = e^{kt}$$

$$\ln(\tfrac{1}{2}) = kt \qquad t = \frac{-\ln 2}{k}$$

$$\boxed{\text{Half life} = \frac{-\ln 2}{k}}$$

Ex Half-life of cesium-137 is 30 years

Suppose 100 mg sample.

- Find formula for $m(t)$
- How long until 1 mg left.

A: $30 = \frac{-\ln 2}{k} \qquad k = -.02310$

$$m(t) = 100e^{-.02310t}$$

$$1 = 100e^{-.02310t}$$

$$.01 = e^{-.02310t}$$

$$-4.61 = -.02310t \qquad t = 199.35 \text{ years}$$