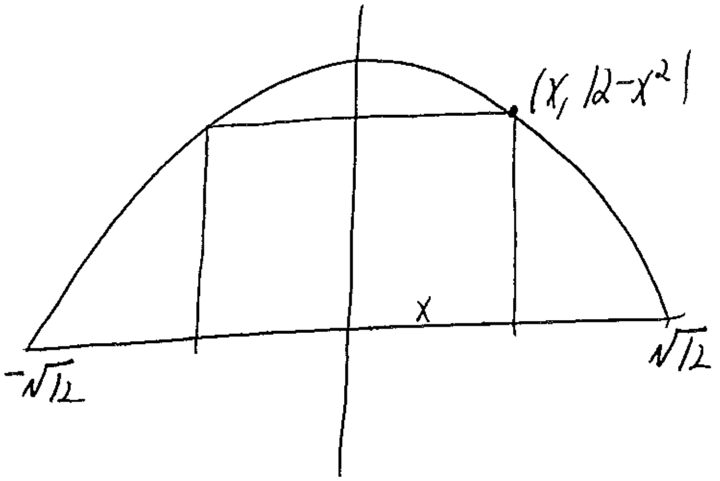
SOLUTIONS

Math 141A- Midterm Exam #3 - November 17, 2014

1. **(20 points)** A rectangle has its base on the x axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have and what are its dimensions?



$$\text{Area} = A(x) = 2x(12-x^2)$$
$$= 24x - 2x^3 \qquad 0 \le x \le \sqrt{12}$$

$$A'(x) = 24 - 6x^2$$

set = 0 $\qquad 24 - 6x^2 = 0$

$x^2 = 4$

$x = \pm 2$

x	$A(x)$
0	0
2	32
$\sqrt{12}$	0

$$\boxed{\begin{array}{c}\text{Max area} = 32 \\ \text{dimensions are } 4 \times 8\end{array}}$$

2. **(10 points)** Suppose $f'(x) = \frac{1}{x} + \sin x$, for $x > 0$. Suppose also that $f(1) = 2$. Find $f(x)$.

$$f(x) = \ln x - \cos x + C$$

$$f(1) = 2 = \ln 1 - \cos 1 + C$$

$$2 = 0 - \cos 1 + C$$

$$C = 2 + \cos 1$$

$$\boxed{f(x) = \ln x - \cos x + 2 + \cos 1}$$

3. **(10 points)** Find

$$\lim_{x\to 1^+} x^{1/(1-x)}.$$

type 1^{∞}

$$y = x^{1/1-x}$$

$$\ln y = \frac{\ln x}{1-x}$$

$$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{1-x}$$ type $\frac{0}{0}$ use LHR

$$\overset{\text{LHR}}{=} \lim_{x\to 1^+} \frac{1/x}{-1} = -1$$

So $\lim_{x\to 1^+} \ln y = -1 \Rightarrow \lim_{x\to 1^+} y = e^{-1}$

$$= \boxed{1/e}$$

4. **(15 points)** Let $f(x) = x^4 - 4x^3 + 10$. Find the critical values and classify them as local max, min, or neither using the **second derivative test**. If the second derivative test fails, you may use the first derivative test.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

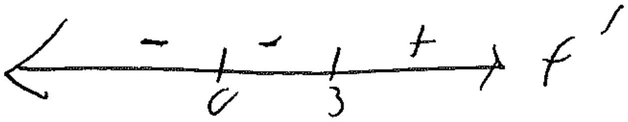
crit points $x = 0, 3$

$f'' = 12x^2 - 24x$

$f''(0) = 0$ No info

$f''(3) = 46 > 0$ so local min

Use 1st der test for $x = 0$



so Neither max nor min

5. **(20 points)** Let

$$f(x) = \frac{(x+1)^2}{1+x^2}.$$

Using the quotient rule we obtain:

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}, \quad f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}.$$

a. Find all x and y intercepts and any asymptotes.

b. Find the intervals where $f(x)$ is increasing or decreasing and any local maximums or local minimums.

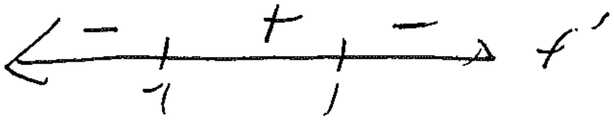
c. Find the intervals where $f(x)$ is concave up or concave down, and determine any inflection points.

d. Neatly sketch the graph of $y = f(x)$, Label the x and y coordinates of any intercepts, local extrema and inflection points.

a. $\boxed{(-1,0) \quad (0,1)}$

$\lim_{x \to \pm\infty} f(x) = 1$ so $\boxed{\text{horiz asym} \quad y=1}$

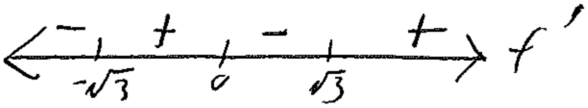
b. $f' = 0$ at $x = \pm 1$



$\boxed{\begin{array}{ll} \text{Decreasing} & (-\infty,-1) \cup (1,\infty) \\ \text{Increasing} & (-1,1) \end{array}}$

$\boxed{\begin{array}{ll} \text{local min} & (-1,0) \\ \text{local max} & (1,2) \end{array}}$

c. $f''=0$ at $0, \sqrt{3}, -\sqrt{3}$



$$\boxed{\begin{array}{l}\text{concave down } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \\ \text{concave up } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)\end{array}}$$

$f(\sqrt{3}) = \frac{(1+\sqrt{3})^2}{4}$

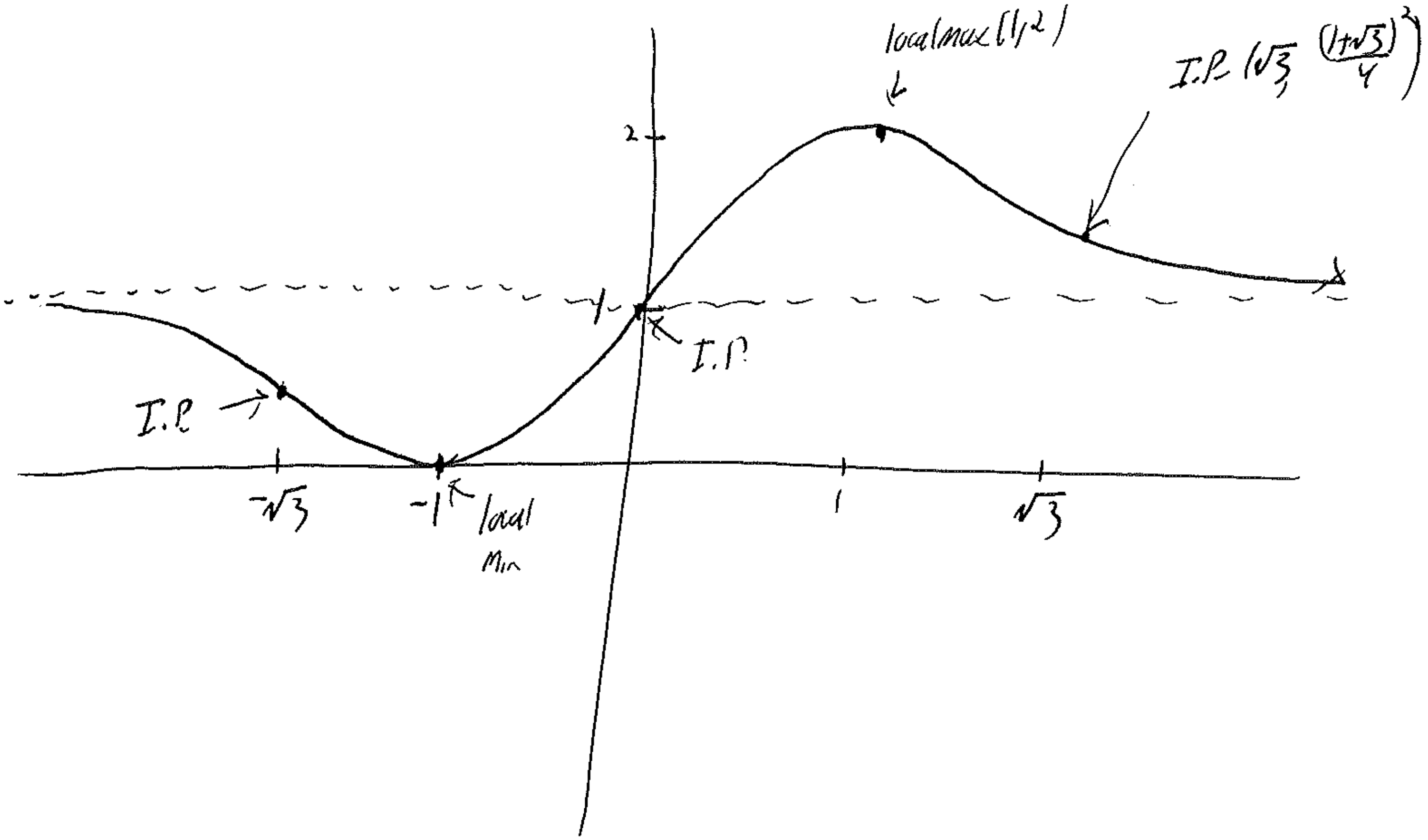
$f(-\sqrt{3}) = \frac{(1-\sqrt{3})^2}{4}$

I.P. $\left(\sqrt{3}, \frac{(1+\sqrt{3})^3}{4}\right)$

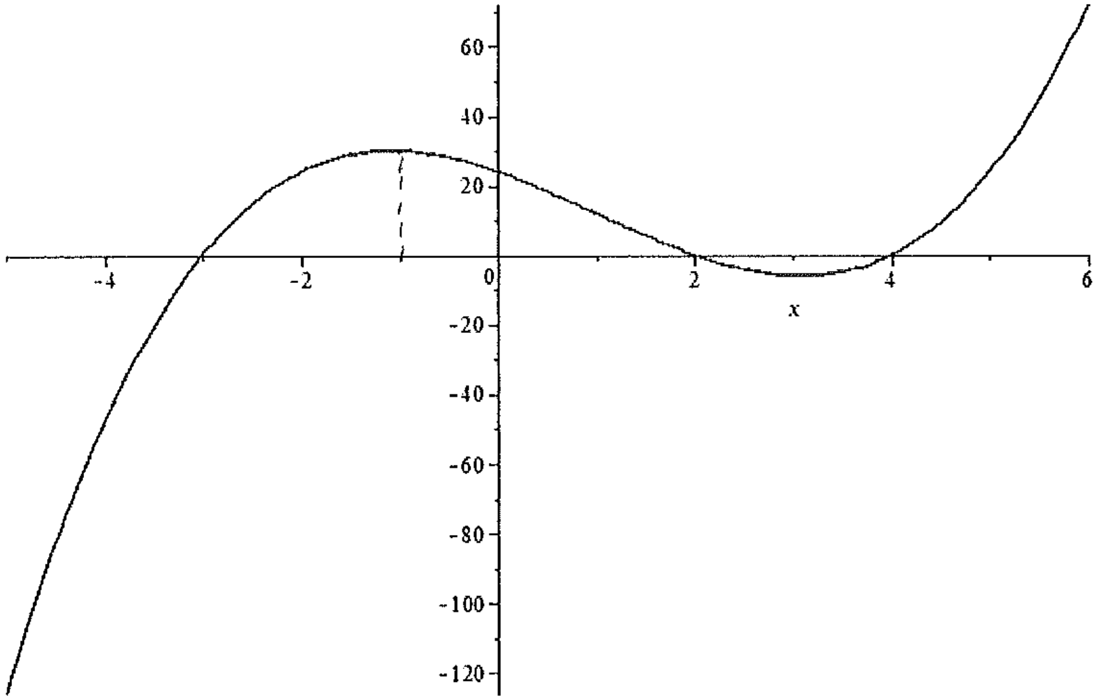
$\left(-\sqrt{3}, \frac{(1-\sqrt{3})^3}{4}\right)$

$(0, 1)$

d.



6. **(15 points)** Below is the graph of $y = f'(x)$. Find the intervals where the original function $f(x)$ is increasing/decreasing and concave up/down.



increasing $(-3, 2) \cup (4, \infty)$ (since $f' \geq 0$)

decreasing $(-\infty, -4) \cup (2, 4)$ " " ≤ 0

concave up where $f'' > 0$ so f' increasing

i.e. $(-\infty, -1) \cup (3, \infty)$

concave down $(-1, 3)$

7. **(10 points)** Prove carefully that $x^4 + 3x + 1$ has exactly one root in the interval $[-2, -1]$. Make sure to cite any theorems that you use.

$f(x) = y^4 + 3x + 1$ is continuous on $[-2, -1]$

$f(-2) = 11$

$f(-1) = -1$ so by Int. Value thm, since $-1 < 0 < 11$,

there is a root in $(-2, -1)$

Now $f'(x) = 4x^3 + 3$

If $f(x)$ had two roots in $[-2, -1]$, by Rolles thm we would have $f'(x) = 0$ for some x in $[-2, -1]$

But $0 = 4x^3 + 3 \Rightarrow x^3 = -3/4$

$\Rightarrow x = \sqrt[3]{-3/4}$

has one root and it is not in the interval

So only one root!