Name:

Math 141- Final Exam - December 14, 2007

Instructions: The exam is worth 150 points. Calculators are not permitted.

1. **(15 points)** Evaluate the following indefinite integrals:

a. $\int e^x \sin(e^x)\, dx$ $\quad u = e^x$

$du = e^x dx$

$\int \sin(u)\, du \quad \int \cos u \rightarrow \boxed{-\cos(e^x)+C}$

b. $\int \sqrt{x} + \frac{1}{x}\, dx \quad = \int \frac{x\sqrt{x}+1}{x}\, dx$

$$\boxed{\frac{2(x)^{3/2}}{3} + \ln x + C}$$

c. $\int \frac{1}{1+x^2}\, dx \rightarrow$ obvious trig antidx

$$\boxed{\tan^{-1}x + C}$$

d. $\int \frac{x}{1+x^2}\, dx \quad u = 1 + x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int \frac{1}{u}\left(du \tfrac{1}{2}\right) \rightarrow \int \frac{\ln u}{2} \rightarrow \boxed{\frac{\ln(1+x^2)}{2} + C}$$

no abs value needed because even function never yields a neg #.

2. **(10 points)** Consider the definite integral $\int_1^3 2x+1\,dx$.

a. Estimate it with a Riemann sum with 6 equal intervals and the right hand endpoints.
b. Write the Riemann sum corresponding to n equal intervals, again using right endpoints.
c. Let $n \to \infty$ to get the actual value of the integral.

You may use the formula
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

d. Check your answer by evaluating the integral with the fundamental theorem of calculus.

☆a.) $R_6 = \sum_{i=1}^{6} f(x_i)\Delta x$, $[\overset{a}{1},\overset{b}{3}] \to \Delta x = \frac{b-a}{n}$ $\Delta x = \frac{1}{3}$

$\frac{1}{3} \cdot \left[f(\frac{4}{3}) + f(\frac{5}{3}) + f(2) + f(\frac{7}{3}) + f(\frac{8}{3}) + f(3) \right]$

$\frac{1}{3} \cdot [\text{ans}] = \boxed{\frac{32}{3}}$

☆b) $R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$ $\quad \Delta x = \frac{2}{n}$

$= \frac{2}{n} \cdot [f(x_1) + f(x_2) \cdots\cdots + f(x_n)]$

☆c.) $x_i = (a + i\Delta x)$ & $\Delta x = \frac{2}{n}$ $\cdots$ so $\lim_{n\to\infty} \sum_{i=1}^{n} (2(i\Delta x + 1) + 1) \cdot \Delta x$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x) \cdot \Delta x$ $\quad \lim_{n\to\infty} \sum_{i=1}^{n} 2i(\Delta x)^2 + 3\Delta x = 2(\Delta x)^2 \sum i + 3\Delta x \sum 1$

$\lim_{n\to\infty} \frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \lim_{n\to\infty} \frac{6}{n} \cdot n = \lim_{n\to\infty} \frac{8n^2+8n}{2n^2} + \lim_{n\to\infty} 6 = 4 + 6 = \boxed{\lim_{n\to\infty} 10}$

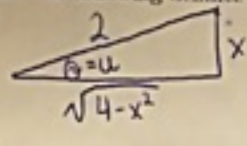
☆d.) $\int_1^3 2x+1\,dx$ [if $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$]

$F(x) = x^2 + x \longrightarrow F(3) - F(1) = \boxed{10}$ ✓

This is an even function so only $\int_0^2 f(x)dx$ is needed. x ans by 2.

3. **(10 points)** Evaluate the following definite integrals by any means you wish (i.e. using FTOC or areas etc...):

a. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$



$\sin^{-1}x = \cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

$\sin\theta = \frac{x}{2}$ $\quad \theta = \sin^{-1}(\frac{x}{2})$

$x = 2\sin(u)dx \quad dx = 2\cos(u)du$

$\sqrt{4-(2\sin u)^2} \cdot \cos u\, du$

$\sqrt{4} \cdot \sqrt{1-\sin^2 u} \cdot \cos u\, du$

$2 \cdot \int \sqrt{\cos^2 u} \cdot \sqrt{4} \cdot \cos(u)\, du = 2 \cdot 2 \int \cos^2(u)\, du \rightarrow$ Trig ID $\cos^2 x = \frac{1+\cos(2x)}{2}$

$\frac{1}{2} \cdot 2 \cdot 2 \left[\int 1 du + \int \cos(2u)\, du \right] \rightarrow v = 2u \rightarrow dv = 2du$, $\frac{1}{2}dv = du$

$\frac{1}{2} \cdot 2 \cdot 2\left(u + \frac{1}{2}\sin(2u)\right)$ subu back in $\rightarrow 2 \cdot \int_0^2 \sqrt{4-x^2}\,dx = 2\left(\sin^{-1}(\frac{x}{2}) + \frac{1}{2} \cdot \sin\left(2\sin^{-1}(\frac{x}{2})\right)\right)$

now plug in $[x=2] = \frac{\pi}{2} + \frac{1}{2} \cdot \sin(2 \cdot \frac{\pi}{2}) = 2 \cdot \frac{\pi}{2} = \pi \cdot 2 = \boxed{2\pi}$

b. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx \qquad u = \ln x, \quad du = \frac{1}{x}dx$

$= \frac{1}{\sqrt{u}} = (u)^{-1/2}du = \frac{\sqrt{u}}{\frac{1}{2}} = \left[2\sqrt{\ln x}\right]_e^{e^4} = 2\sqrt{4} - 2\sqrt{1} = \boxed{2}$

c. $\int_0^5 x(x^2+1)^{15}dx$

$\int_0^5 (u)^{15}[\frac{1}{2}du] \qquad u = x^2+1, \quad du = 2x\,dx, \quad \frac{1}{2}du = x\,dx$

$\frac{(u)^{16}}{16 \cdot 2} = \left[\frac{(x^2+1)^{16}}{32}\right]_0^5 = \boxed{\frac{(26)^{16}}{32} - \frac{1}{32}}$

4. (10 points) Let

$$f(x) = \left(\int_1^x \frac{t}{1+t+t^2}\,dt.\right)\frac{d}{dx}$$

a. What is $f'(x)$.

b. On which intervals is $f(x)$ increasing/ decreasing?

a.) $f'(x) = \dfrac{x}{1+x+x^2}$

b) $-$ $+$ $f'(x)$

inflection inflection $f''(x)$

$-$ -1 $+$ 0 $+$ 1 $-$

$\boxed{\begin{array}{l}\text{increase}: (-\infty, 0) \\ \text{decrease}: (0, \infty)\end{array}}$

5. (5 points) The velocity function of a particle moving along a line is given in meters per second by $v(t) = 3t - 5$ for $0 \le t \le 3$. Find the total distance the particle traveled during the time interval.

$f'(x) = 3x - 5 = 0 \quad x = \frac{5}{3}$ $-$ $\frac{5}{3}$ $+$

$$f(x) = \int_0^3 3t - 5\,dt$$

$$\text{distance} = -\int_0^{5/3} g(t)\,dt + \int_{5/3}^{3} g(t)\,dt$$

$$-\left[\frac{3x^2}{2} - 5x\right]_0^{\frac{5}{3}} + \left[\frac{3x^2}{2} - 5x\right]_{\frac{5}{3}}^{3} = \boxed{\frac{41}{6}\text{ meters}}$$

6. (5 points) If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t minutes, what does $\int_0^{60} r(t)dt$ represent?

The amount of oil leaked in gallons in the first hour.

7. (10 points) Find the area under the graph of $y = \sin(2x)$ and above the interval $[0, \pi/2]$ on the x axis.

$u = 2x \quad du = 2dx \quad \frac{1}{2}du = dx$

$$\int_0^{\frac{\pi}{2}} \sin(2x)\,dx \quad = \left[\frac{-\cos(2x)}{2}\right]_0^{\frac{\pi}{2}}$$

$$\frac{-\cos(\frac{\pi}{2})}{2} - \frac{-\cos(0)}{2} = \boxed{1}$$

8. (5 points) State precisely the intermediate value theorem, including any necessary hypotheses.

Let the function f(x) be continuos on [a, b] there exists atleast on value c such that f(c) = D, D being a point on the y axis between f(a) & f(b).

9. (10 points) Find the equation of the tangent line to the graph of $y = x^2 + 2x + 1$ at the point where $x = 2$.

$f'(x) = 2x + 2 \rightarrow f'(2) = 6 = m$

$f(2) = 4 + 4 + 1 = 9$ $(2, 9)$. slope: 6

$y - y_1 = m(x - x_1)$

$\boxed{y - 9 = 6(x - 2)}$

10. (5 points) Sketch the graph of a function which is continuous but not differentiable at $x = 2$.

11. **(20 points)** Find $\frac{dy}{dx}$

a. $y = \overset{f}{x} \overset{g}{\cos}(x)$

$PR = fg' + gf'$

$\boxed{y' = x(-\sin(x)) + \cos(x)}$

b. $y = \frac{x}{x^2+1}$ f g

$QR = \frac{gf' - fg'}{g^2}$

$\boxed{\frac{(x^2+1) - x(2x)}{(x^2+1)^2}}$

f(x) c. $y = \int_1^x \sqrt{t^2 + \cos t}\, dt$

by FTOC (Pt #1):

$\boxed{f'(x) = \sqrt{x^2 + \cos(x)}}$

d. $y = \tan(e^{2x})$

Chain Rule

$\boxed{\sec^2(e^{2x}) \cdot e^{2x} \cdot 2}$

e. $\ln(y) + \overset{f}{x}\overset{g}{y} = 3$

Implicit Differentiation

$y \cdot \left(\frac{1}{y} y' + xy' + y \right) = 0 \cdot y$

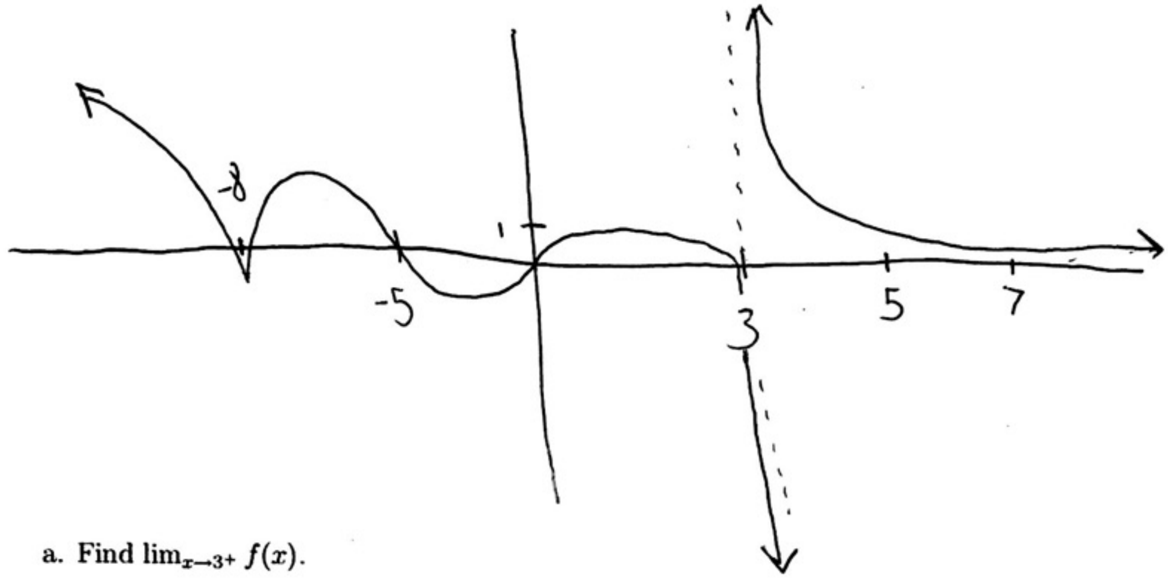
$y' + xy' + y^2 = 0$

$y' + xy' = -y^2$

$y'(1 + x) = -y^2$

$\boxed{y' = \frac{-y^2}{1+xy}}$

12. **(15 points)** Below is sketched the graph of $y = f(x)$. Answer the following questions.



a. Find $\lim_{x\to 3^+} f(x)$.

∞

b. Estimate $f'(5)$.

$f'(5) = \frac{1}{2}$

d. Estimate the location of any inflection points.

@ $x = 5, 0, 3$

e. At what x values does $f(x)$ fail to be differentiable?

@ -8 $\cancel{3}$ 3

f. Estimate $\int_0^3 f(x)\,dx$.

3

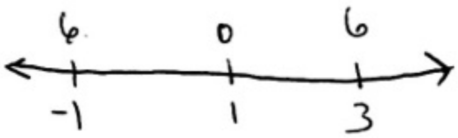
g. Find $\lim_{x\to 7} f(x)$

$\frac{3}{4}$

13. **(10 points)** Let $f(x) = x^2 - 2x + 3$. Find the global maximum and minimum values of $f(x)$ on the interval $[-1, 3]$.

$f'(x) = 2x - 2 = 0$

$x = 1$



Global max @ $y = 6$
Global min @ $y = 0$
on $[-1, 3]$

14. **(10 points)** Evaluate the following limits, if they exist:

$a.\ \lim_{x\to\infty} \dfrac{x^3+x+1}{x^4+2x+3}$ $\quad b.\ \lim_{x\to 3^+} \dfrac{x^2-9}{x^2-2x-3}$ $\frac{0}{0}$ $\quad c.\ \lim_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h}$. $\frac{0}{0}$

a.

$$\frac{\dfrac{x^3}{x^3} \cdots}{\dfrac{x^{4}}{x^3} \cdots}$$

$\dfrac{1}{\text{large \#}}$

$\lim_{x\to\infty} f(x) = 0$

b.

L'hop $\dfrac{2x}{2x-2} = \dfrac{6}{4}$

$\lim_{x\to 3^+} = \dfrac{3}{2}$

c.

$$\frac{(16+x)^{1/4} - 2}{x}$$

L'hop $\dfrac{\dfrac{(16+x)^{-3/4}}{4}}{1}$

simplify:

$$\frac{1}{4(16+x)^{3/4}}$$

plug $x = 0$

$\dfrac{1}{4(8)} = \lim_{h\to 0} f(h) = \dfrac{1}{32}$

15. **(10 points)** Let $f(x) = 4x^3 + 3x^2 - 6x + 1$. Find the intervals on which f is increasing or decreasing. Find the local maximum and minimum values of f. Find the intervals of concavity and inflection points. Then neatly sketch the graph $y = f(x)$.

$f'(x) = 12x^2 + 6x - 6$ $\quad 6(2x^2 + x - 1) = 0 \quad (2x-1)(x+1) = 0$

$x = \frac{1}{2} \quad x = -1$

$+ \quad -1 \quad - \quad 1/2 \quad +$

$\boxed{\text{inc}: (-\infty, -1) \cup (\tfrac{1}{2}, \infty) \quad \text{dec}: (-1, 1/2)}$

$\boxed{\text{local max}: @\ x = -1 \quad \text{local min}: @\ x = \tfrac{1}{2}}$

$f''(x) = 24x + 6 = 0 \quad x = \frac{-6}{24} = \frac{-1}{4}$

$- \quad \frac{-1}{4} \quad +$

$\boxed{\text{inflection point}: @\ x = \tfrac{-1}{4} \quad \text{concave down}: (-\infty, -\tfrac{1}{4}) \quad \text{concave up}: (-\tfrac{1}{4}, \infty)}$