Name: SOLUTIONS

Math 1830- Final Exam - December 14, 2006

Instructions: The exam is worth 150 points. Calculators are not permitted.

1. **(20 points)** Evaluate the following indefinite integrals:

a. $\int t^3 + t - 5\, dt$

$\frac{1}{4}t^4 + \frac{1}{2}t^2 - 5t + C$

b. $\int x\sqrt{1+2x^2}\, dx$

$u = 1+2x^2 \quad du = 4x\, dx$

$\int \frac{1}{4} u^{1/2}\, du = \frac{2}{3}\cdot\frac{1}{4}u^{3/2} + C = \frac{1}{6}(1+2x^2)^{3/2} + C$

c. $\int \sec x \tan x\, dx$

$\sec x + C$

d. $\int 3 + 2x(x^2-2)^5\, dx$

$u = x^2 - 2 \quad du = 2x\, dx$

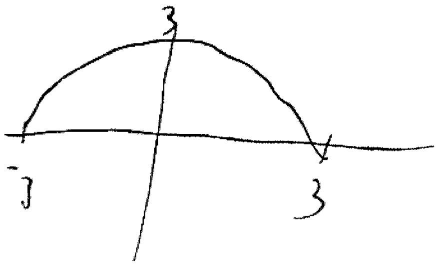
$3x + \frac{1}{6}(x^2-2)^6 + C$

e. $\int 6\, dx$.

$6x + C$

2. **(15 points)** Evaluate the following definite integrals by any means you wish:

a. $\int_{-3}^{3} \sqrt{9 - v^2}\, dv$



$$\text{Area} = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi$$

b. $\int_2^3 t(3-t)^{1/3}\, dt$

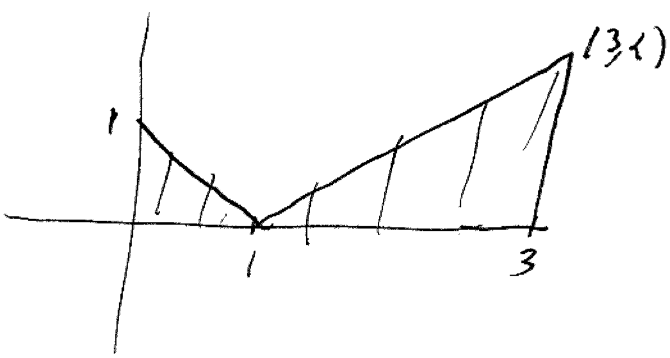
$u = 3 - t \quad du = -dt \quad t = 3 - u$

$$= \int_1^0 -(3-u)u^{1/3}\, du = \int_1^0 -3u^{1/3} + u^{4/3} = \frac{-9}{4}u^{4/3} + \frac{3}{7}u^{7/3}\Big|_1^0$$

$$= 0 - \left(-\frac{9}{4} + 3/7\right)$$

$$= 9/4 - 3/7$$

c. $\int_0^3 |\, x - 1 \,|\; dx$



$$\text{Area} = \frac{1}{2}\cdot 1 \cdot 1 + \frac{1}{2}\cdot 2 \cdot 2$$

$$= \frac{1}{2} + 2$$

$$= 5/2$$

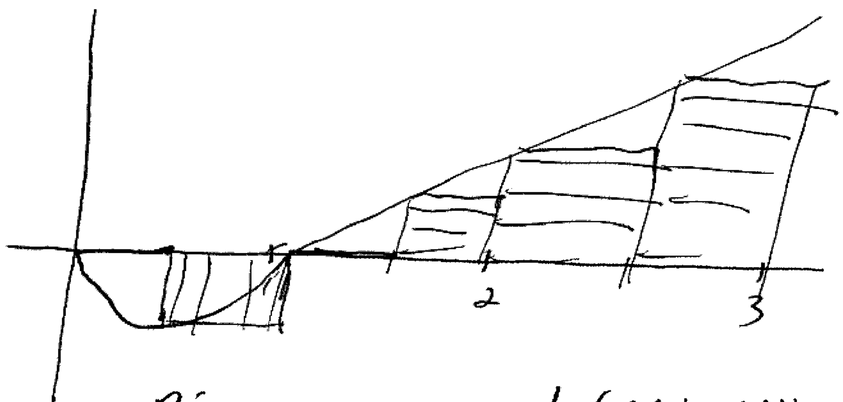
3. **(10 points)** If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)\, dt$ represent?

total oil which leaks in first 2 hours

4. **(15 points)** Evaluate the Riemann sum for

$$f(x) = x^2 - x \quad 0 \le x \le 3$$

with six equal intervals and taking your sample points to be the left endpoint of each interval. Explain, with the aid of a diagram, what the Riemann sum represents.



It is area
minus area

Rieman sum is $\frac{1}{2}(f(0)+f(\frac{1}{2})+f(1)+f(\frac{3}{2})+f(2)+f(\frac{5}{2}))$

$= \frac{1}{2}(0 - \frac{1}{4} + 0 + \frac{3}{4} + 2 + \frac{15}{4}) = \frac{1}{2}(\frac{25}{4}) = 25/8$

5. **(15 points)**

a. Find the area under the graph of $y = x^2 + 2$ and above the interval $[1, 2]$ on the x axis.

$\int_1^2 x^2+2\,dx = \frac{x^3}{3}+2x\Big|_1^2 = (\frac{8}{3}+4)-(\frac{1}{3}+2)$

$= \frac{7}{3}+2 = 16/3$

b. Let $f(x) = x^2$. Find the average value of f on the interval $[2, 5]$. Then find a value $c \in [2, 5]$ such that $f_{ave} = f(c)$.

$f_{ave} = \frac{1}{5-2}\int_2^5 x^2\,dx = \frac{1}{3}\cdot\frac{1}{3}x^3\Big|_2^5 = \frac{1}{9}(125-8)$

$= \frac{117}{9}$

$x^2 = \frac{117}{9} \quad x = \frac{\sqrt{117}}{3}$

$c = \frac{\sqrt{117}}{3}$

6. **(20 points)** You are given $g(x)$. Find the derivative $g'(x)$:

a. $g(x) = x\sin(x)$

$\sin x + x\cos x$

b. $g(x) = \frac{\sin x}{x^2+1}$

$$\frac{(x^2+1)\cos x - (\sin x)(2x)}{(x^2+1)^2}$$

c. $g(x) = \int_1^x \sqrt{t^2+\cos t}\,dt$

$\sqrt{x^2+\cos x}$

d. $g(x) = \int_1^{1/x} t^2 + t^3\,dt$

$h(u) = \int_1^u t^2+t^3\,dt \quad h'(u) = u^2+u^3$

$g(x) = h\left(\frac{1}{x}\right)$

$g' = h'\left(\frac{1}{x}\right)\cdot \frac{-1}{x^2} = -\frac{1}{x^2}\left(\frac{1}{x^2}+\frac{1}{x^3}\right)$

e. $g(x) = |x|$

$$g'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{DNE} & x = 0 \end{cases}$$

7. **(10 points)** Use implicit differentiation to find the equation of the tangent line to the ellipse $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.

$$2x + y + xy' + 2yy' = 0$$

$$2 + 1 + y' + 2y' = 0$$

$$y' = -1$$

$$y - 1 = -1(x - 1)$$

8. **(10 points)** Evaluate:

a. $\lim_{t\to 0} \frac{\sin(4t)}{t}$

$$4$$

b. $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81} + \frac{2}{243} \cdots$

$$a = \frac{2}{3} \quad r = -1/3$$

$$\frac{2/3}{1 - -1/3} = \frac{2/3}{4/3} = 1/2$$

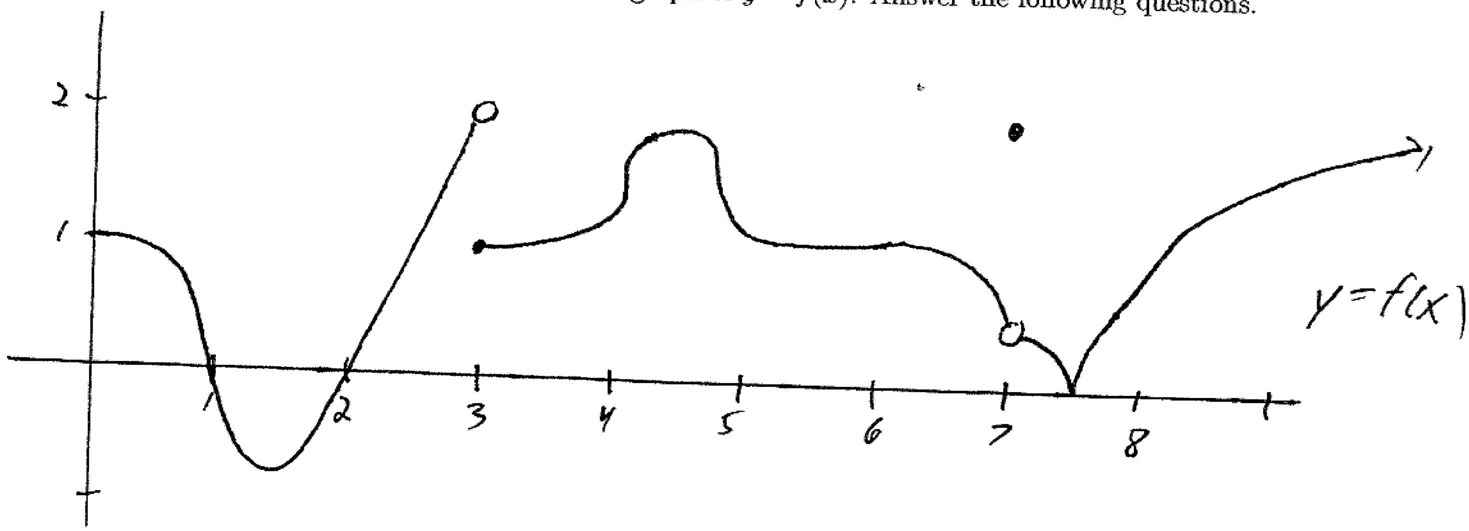
c. $\lim_{n\to\infty} \sum_{i=1}^{n} \sin(\frac{\pi i}{2n}) \frac{\pi}{2n}$

$$= \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 0 - (-1) = 1$$

d. $\lim_{x\to\infty} \frac{x^3+2x-1}{3x^4+6x^2-5x+12}$

$$0$$

9. **(15 points)** Below is sketched the graph of $y = f(x)$. Answer the following questions.



a. Find $\lim_{x \to 3^+} f(x)$.

1

b. Estimate $f'(5)$.

$-1/2$

c. Estimate $f''(5)$.

1

d. Estimate the location of any inflection points.

$x = 1,\ x = 4,\ x = 4.7\ \ x = 6$

e. At what x values does $f(x)$ fail to be differentiable?

$x = 3, 4, 7, 7.5$

f. Estimate $\int_0^3 f(x)\, dx$.

1

g. Find $\lim_{x \to 7} f(x)$

$1/2$

10. **(10 points)** *Using the definition*, show that the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} 2x+h = 2x$$

11. **(10 points)** Prove that the equation $3 + x + 6x^3 + x^7 = 0$ has exactly one real root. Be clear about which theorems you cite in your proof, and why they apply.

Let $f(x) = x^7+6x^3+x+3$. $f(x)$ is continuous and differentiable on $(-\infty,\infty)$.

$f(-1) = -5$ $f(0) = 3$ Thus by the intermediate value theorem $f(c) = 0$ for some c in $(-1,0)$, so $f(x)$ has a root.

Suppose it has 2 roots so $f(r_1) = f(r_2) = 0$, some $r_1 < r_2$. By Rolles Thm, there is a c in (r_1, r_2) with $f'(c) = 0$. But $f'(x) = 7x^6+18x^2+1$ is never 0, so this can't happen, it has exactly one root.