Name: SOLUTIONS

Math 141- Final Exam - December 14, 2007

Instructions: The exam is worth 150 points. Calculators are not permitted.

1. **(15 points)** Evaluate the following indefinite integrals:

a. $\int e^x \sin(e^x)\ dx$

$u = e^x \quad du = e^x dx$

$\int \sin u\, du = -\cos u + C = -\cos(e^x) + C$

b. $\int \sqrt{x} + \frac{1}{x}\ dx$

$\boxed{\frac{2}{3}x^{\frac{3}{2}} + \ln|x| + C}$

c. $\int \frac{1}{1+x^2}\ dx$

$\boxed{\tan^{-1}x + C}$

d. $\int \frac{x}{1+x^2}\ dx$

$u = 1+x^2 \quad du = 2x\, dx$

$\int \frac{1}{2} \cdot \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|1+x^2| + C$

2. **(10 points)** Consider the definite integral $\int_1^3 2x+1\,dx$.

a. Estimate it with a Riemann sum with 6 equal intervals and the right hand endpoints.
b. Write the Riemann sum corresponding to n equal intervals, again using right endpoints.
c. Let $n \to \infty$ to get the actual value of the integral.

You may use the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

d. Check your answer by evaluating the integral with the fundamental theorem of calculus.

a. $\Delta x = \frac{3-1}{6} = 1/3$

$R_6 = \frac{1}{3}\left(f(4/3) + f(5/3) + f(6/3) + f(7/3) + f(8/3) + f(9/3)\right)$

$= \frac{1}{3}\left(11/3 + 13/3 + 15/3 + 17/3 + 19/3 + 21/3\right) = \frac{1}{3} \cdot \left(93/3\right) = 31/3$

b $\quad x_i = 1 + 2/n \cdot i$ so $\quad \sum_{i=1}^{n} \left(2\left(1 + 2i/n\right) + 1\right)\left(2/n\right)$

c $\quad \sum_{i=1}^{n} \left(2 + \frac{4i}{n} + 1\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{6}{n} + \frac{8i}{n^2}$

$= \frac{6}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i$

$= \frac{6}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} = 6 + \frac{4n(n+1)}{n^2}$

Letting $n \to \infty$ we get $6 + 4 = 10$

d $\quad \int_1^3 2x+1 = x^2+2\Big|_1^3 = 12 - 2 = 10$

4. **(10 points)** Let

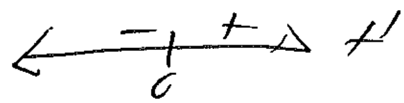
$$f(x) = \int_1^x \frac{t}{1+t+t^2}\, dt.$$

a. What is $f'(x)$.

b. On which intervals is $f(x)$ increasing/ decreasing?

a. $f'(x) = \dfrac{x}{1+x+x^2}$

b $1+x+x^2$ is always >0 so

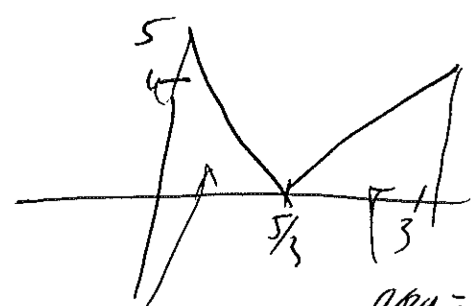


increasing $(0,\infty)$
dec $(-\infty, 0)$

5. **(5 points)** The velocity function of a particle moving along a line is given in meters per second by $v(t) = 3t - 5$ for $0 \le t \le 3$. Find the total distance the particle traveled during the time interval.

speed $= |v(t)| = |3t-5|$

distance $= \int_0^3 |3t-5|\, dt$



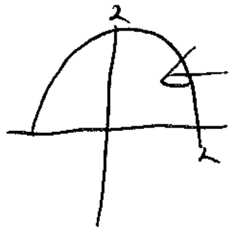
area $= \dfrac{5 \cdot 5/3}{2}$

area $= \dfrac{4 \cdot 4/3}{2}$

total area $= 25/6 + 16/6 = 41/6$

3. **(10 points)** Evaluate the following definite integrals by any means you wish (i.e. using FTOC or areas etc...):

a. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$



$\text{area} = \frac{1}{2}\pi r^2 = 2\pi$

b. $\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$u = \ln x \quad du = 1/x\,dx$

$\int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x}\,\Big|_{x=e}^{x=e^4}$

$= 2\sqrt{4} - 2\sqrt{1} = 2$

c. $\int_0^5 x(x^2+1)^{15}dx$

$u = (x^2+1) \quad du = 2x\,dx$

$x=0 \to u=1$

$x=5 \to u=26$

$= \int_1^{26} \frac{1}{2}u^{15}\,du = \frac{1}{32}u^{16}\Big|_1^{26} =$

$\frac{1}{32}(26^{16}-1)$

6. **(5 points)** If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t minutes, what does $\int_0^{60} r(t)dt$ represent?

total leakage from $t=0$ to $t=60$

7. **(10 points)** Find the area under the graph of $y = \sin(2x)$ and above the interval $[0, \pi/2]$ on the x axis.

$$\int_0^{\pi/2} \sin 2x \, dx = -\cos(2x) \cdot \frac{1}{2} \Big|_{x=0}^{x=\pi/2}$$

$$= -\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0$$

$$= \frac{-1}{2}(-1) + \frac{1}{2} = 1$$

8. **(5 points)** State precisely the intermediate value theorem, including any necessary hypotheses.

Let $f(x)$ be continuous on $[a,b]$

Suppose C lies between $f(a)$ and $f(b)$.

Then there exists a d in (a,b)

so $f(d) = C$.

9. **(10 points)** Find the equation of the tangent line to the graph of $y = x^2 + 2x + 1$ at the point where $x = 2$.

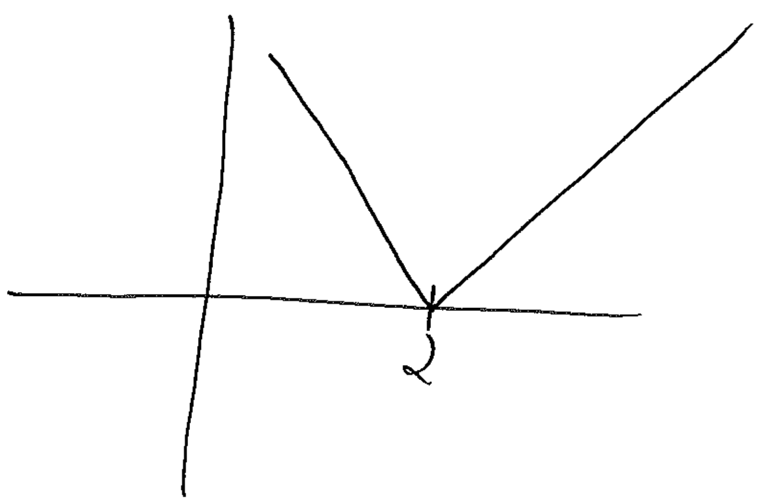
$y' = 2x + 2$

slope $= y'(2) = 6$

point $= (2, 2^2 + 4 + 1) = (2, 9)$

$y - 9 = 6(x - 2)$

10. **(5 points)** Sketch the graph of a function which is continuous but not differentiable at $x = 2$.



11. **(20 points)** Find $\frac{dy}{dx}$

a. $y = x\cos(x)$

$$\boxed{\cos x - x\sin x}$$

b. $y = \frac{x}{x^2+1}$

$$\frac{x^2+1-2x^3}{(x^2+1)^2} = \boxed{\frac{1-x^3}{(x^2+1)^2}}$$

c. $y = \int_1^x \sqrt{t^2+\cos t}\,dt$

$$\boxed{\sqrt{x^2+\cos x}}$$

d. $y = \tan(e^{2x})$

$$\sec^2(e^{2x}) \cdot e^{2x} \cdot 2$$

e. $\ln(y) + xy = 3$

$$\frac{1}{y}y' + y + xy' = 0$$

$$y' = \frac{-y}{1/y + x} = \frac{-y^3}{1+xy}$$

12. **(15 points)** Below is sketched the graph of $y = f(x)$. Answer the following questions.

a. Find $\lim_{x \to 3+} f(x)$.

b. Estimate $f'(5)$.

d. Estimate the location of any inflection points.

e. At what x values does $f(x)$ fail to be differentiable?

f. Estimate $\int_0^3 f(x)\,dx$.

g. Find $\lim_{x \to 7} f(x)$

13. **(10 points)** Let $f(x) = x^2 - 2x + 3$. Find the global maximum and minimum values of $f(x)$ on the interval $[-1, 3]$.

$f' = 2x - 2$ set $= 0$

crit point $x = 1$

x	$f(x)$
-1	6
1	2
3	6

max value 6

min value 2

14. **(10 points)** Evaluate the following limits, if they exist.

$$a.\ \lim_{x\to\infty} \frac{x^3+x+1}{x^4+2x+3} \qquad b.\ \lim_{x\to 3^+} \frac{x^2-9}{x^2-2x-3} \qquad c.\ \lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h}.$$

$=$

$$\lim_{x\to\infty} \frac{1/x + 1/x^3 + 1/x^4}{1 + 2/x^3 + 3/x^4}$$

$=$

0

b $$= \lim_{x\to 3^+} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+1)} = 6/4 = 3/2$$

c. If ~~for~~ $f(x) = \sqrt[4]{x}$ this is $f'(16)$

$$f'(x) = \frac{1}{4} x^{-3/4}$$

$$f'(16) = \frac{1}{4\sqrt[4]{16}^{\,3}} = 1/32$$

15. **(10 points)** Let $f(x) = 4x^3 + 3x^2 - 6x + 1$. Find the intervals on which f is increasing or decreasing. Find the local maximum and minimum values of f. Find the intervals of concavity and inflection points. Then neatly sketch the graph $y = f(x)$.

$f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x-1)(x+1)$

$+ \quad -1 \quad - \quad 1/2 \quad +$ $\quad f'$

increasing $(-\infty, -1) \cup (1/2, \infty)$

decreasing $(-1, 1/2)$

$1/2 + 3/4 - 3 + 1$

Local max $(-1, 6)$ Local min $(1/2, -3/4)$

$f'' = 24x + 6$

$- \quad -1/4 \quad +$ $\quad f''$

concave up $(-1/4, \infty)$

concave down $(-\infty, -1/4)$

$\frac{-4}{4^3} + 3(1/16) + 6/4 + 1$

$= 1/8 + 3/2 + 1$

I.P $(-1/4, 21/8)$