MTH444/MTH544 solution to midterm exam

1(a) The continuity equation: $\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0$ (1)

Look for curve satisfying $\frac{d}{dt}\rho(X(t),t) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dX}{dt} = 0$ (2)

Compare (2) with (1): $\frac{dX}{dt} = \rho$

$\underset{x}{\underset{\parallel}{X(t)}} - \underset{x_0}{\underset{\parallel}{X(0)}} = \int_0^t \rho\, dt = \rho_0 t$ so $x = x_0 + \rho_0 t$

Now use ρ_0 values from initial condition:

$x_0 \le -1$ $\quad x = x_0 + 1\cdot t = x_0 + t$

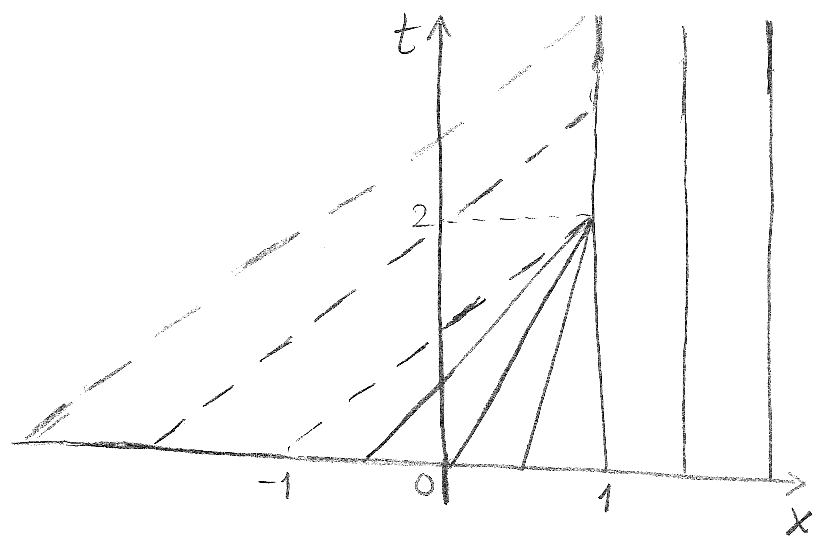
$-1 \le x_0 \le 1$ $\quad x = x_0 + \frac{1}{2}(1-x_0)t$

$x_0 \ge 1$ $\quad x = x_0$

$x_0 = -1$ $\quad x = -1 + t$ $\quad t = x+1$

$x_0 = 0$ $\quad x = \frac{1}{2}t$ $\quad t = 2x$

$x_0 = \frac{1}{2}$ $\quad x = \frac{1}{2} + \frac{1}{4}t$ $\quad t = 4x - 2$



(b) Let $x_L = -1 + t$ the char. for $x_0 = -1$ and $x_R = 1$.

Shock wave develops when $x_L(t_s) = x_R(t_s)$.

$\Rightarrow$ $-1 + t_s = 1$ $\Rightarrow$ $t_s = 2$

The shock forms at $x_s = 1$ $\quad x_L(2) = x_R(2) = x_s = 1$.

$s(t)$ – shock location $\quad s'(t)$ – shock velocity

$$s'(t) = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L} \qquad c(\rho) = \rho = \frac{dJ}{d\rho} \qquad J = \tfrac{1}{2}\rho^2 + K$$

Since $J = \rho v$, $J(0) = 0$, so $J = \frac{1}{2}\rho^2$.

$$s'(t) = \frac{\frac{1}{2}\rho_R^2 - \frac{1}{2}\rho_L^2}{\rho_R - \rho_L} = \frac{0 - \frac{1}{2}}{-1} = \frac{1}{2}$$

$$s(t) = x_s + s'(t)(t - t_s) = 1 + \tfrac{1}{2}(t-2) = \tfrac{1}{2}t$$

(c) solution for $0 \le t \le 2$.

$x_0 \le -1 \Rightarrow x \le -1 + t \quad \rho(x,t) = \rho(x_0, 0) = 1$

$x_0 \ge 1 \Rightarrow x \ge 1 \quad \rho(x,t) = \rho(x_0,0) = 0$

$-1 \le x_0 \le 1 \Rightarrow -1+t \le x \le 1 \quad \rho(x,t) = \rho(x_0,0) = \frac{1}{2}(1 - x_0)$

Express x_0 in terms of x, t.

$$x = x_0 + \tfrac{1}{2}(1 - x_0)t = x_0 + \tfrac{1}{2}t - \tfrac{1}{2}x_0 t$$

$$x - \tfrac{1}{2}t = x_0(1 - \tfrac{1}{2}t) \Rightarrow x_0 = \frac{(x - \frac{1}{2}t)}{(1 - \frac{1}{2}t)} = \frac{2x - t}{2 - t}$$

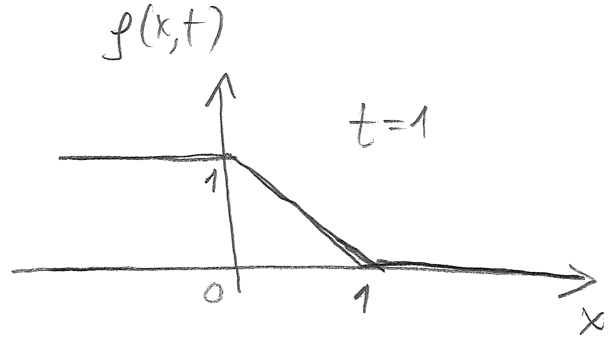
$$\rho(x,t) = \frac{1}{2} - \frac{(2x-t)}{2(2-t)} = \frac{(2-t) - (2x-t)}{2(2-t)} = \frac{\cancel{2}(1-x)}{\cancel{2}(2-t)}$$

$$\rho(x,t) = \begin{cases} 1 & x \le -1+t \\ \frac{1-x}{2-t} & -1+t \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

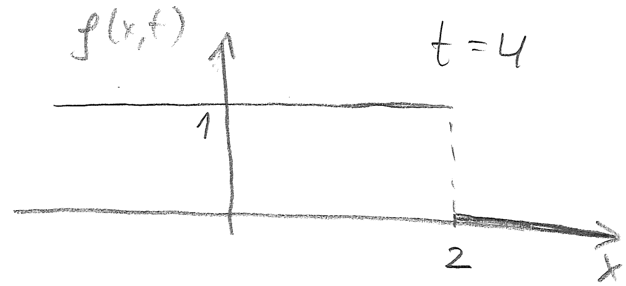
solution for $t > t_s = 2$

$$\rho(x,t) = \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x > \frac{1}{2}t \end{cases}$$

(d) $t=1$ $\quad \rho(x,t) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$



$t=4$ $\quad \rho(x,t) = \begin{cases} 1 & x < 2 \\ 0 & x > 2 \end{cases}$



2(a) The total mass of the cylinder's segment at time t:

$$M = \int_{\alpha(t)}^{\beta(t)} \sigma \rho(x,t)\, dx$$

Since the mass is constant: $\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} \sigma \rho(x,t)\, dx = 0$

By Leibnitz's rule

$$\int_{\alpha(t)}^{\beta(t)} \frac{\partial \rho}{\partial t}\, dx + \rho(\beta,t)\frac{d\beta}{dt} - \rho(\alpha,t)\frac{d\alpha}{dt} = 0$$

$$\frac{d\beta}{dt} = v(\beta,t) \qquad \frac{d\alpha}{dt} = v(\alpha,t) \quad \Rightarrow \quad \int_{\alpha(t)}^{\beta(t)} \frac{\partial \rho}{\partial t}\, dx + \rho(\beta,t)v(\beta,t) - \rho(\alpha,t)v(\alpha,t) = 0$$

$$\int_{\alpha(t)}^{\beta(t)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right] dx = 0$$

for any $t \geq 0$, A_R, A_L st $A_R > A_L$.

It follows: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$ (*)

(b) Rewrite (*) as: $\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$ (1)

Require $R(A,t) = \rho(X(A,t),t)$

Use $\frac{\partial R}{\partial t} = \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}$ (2)

Also $\frac{\partial V}{\partial A} = \frac{\partial v}{\partial x}\left(1 + \frac{\partial \bar{U}}{\partial A}\right)$

$$\frac{\partial V}{\partial A} = \frac{\partial}{\partial A}\frac{\partial \bar{U}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \bar{U}}{\partial A} = \frac{\partial}{\partial t}(1+\bar{U}_A)$$

So $\frac{\partial v}{\partial x} = \frac{1}{(1+\bar{U}_A)}\frac{\partial}{\partial t}(1+\bar{U}_A) \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial}{\partial t}\ln(1+\bar{U}_A)$ (3)

Substitute (2), (3) into (1):

$$\frac{\partial R}{\partial t} + R\frac{\partial}{\partial t}\ln(1+\bar{U}_A) = 0 \quad (4)$$

The continuity equation in material coordinates.

To solve it, rewrite (4) as:

$$\frac{\partial}{\partial t}\ln R = -\frac{\partial}{\partial t}\ln(1+\bar{U}_A)$$

Integrate wrt t:

$$\ln R(A,t) - \ln R(A,0) = -\ln\left[1+\frac{\partial U}{\partial A}(A,t)\right] + \cancel{\ln\left[1+\frac{\partial U}{\partial A}(A,0)\right]}$$

$$U(A,t) = X(A,t) - A \Rightarrow U(A,0)=0 \Rightarrow \frac{\partial U}{\partial A}(A,0)=0$$

$$\ln\left[\frac{R(A,t)}{R(A,0)}\right] = \ln\left[\frac{1}{1+U_A}\right]$$

So the solution is $R(A,t) = \dfrac{R(A,0)}{1+U_A(A,t)}$

(3) (a) Momentum equation in M coordinates in steady state:

$$\frac{\partial T}{\partial A} = -R_o F = -\alpha(1+A/L_0)g$$

BCs : $T(l_0)=0$ (free lower end) $T(0)=0$ (fixed upper end)

Integrate wrt A from A to l_0:

$$\overset{0}{T(l_0)} - T(A) = -\alpha g\int_A^{l_0}\left(1+\frac{A'}{l_0}\right)dA' = -\alpha g\left[A' + \frac{A'^2}{2l_0}\right]_A^{l_0}$$

$$-T(A) = -\alpha g(l_0-A) - \tfrac{1}{2}\alpha g\left(l_0 - \frac{A^2}{l_0}\right)$$

The stress in steady state is:

$$T(A) = \alpha g(l_0-A) + \tfrac{1}{2}\alpha g\left(l_0 - \frac{A^2}{l_0}\right)$$

(b) Find the displacement of l_0, $U(l_0)$.

Since the cord is linearly elastic: $T = E\frac{dU}{dA}$

$$\frac{dU}{dA} = \frac{T(A)}{E} = \frac{\alpha g}{E}(l_0 - A) + \frac{\alpha g}{2E}\left(l_0 - \frac{A^2}{l_0}\right)$$

Integrate wrt A from 0 to A:

$$U(A) - \cancel{U(0)} = \frac{\alpha g}{E}\left(l_0 A - \frac{1}{2}A^2\right) + \frac{\alpha g}{2E}\left(l_0 A - \frac{A^3}{3l_0}\right)$$

$$\Rightarrow \quad U(l_0) = \frac{\alpha g l_0^2}{2E} + \frac{\alpha g l_0^2}{3E} = \frac{5\alpha g l_0^2}{6E}$$

The total length: $l = l_0 + \dfrac{5\alpha g l_0^2}{6E}$